Rate control of a queue with quality-of-service constraint under bounded and unbounded action spaces

by

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DEDICATION

I would like to dedicate this thesis to my mother and father for their unconditional support.

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ABSTRACT

We consider a simple Markovian queue with Poisson arrivals and exponential service times for jobs. The controller can choose service rates from a specified action space depending on number of jobs in the queue. The queue has a finite buffer and when full, new jobs get rejected. The controller's objective is to choose optimal (state-dependent) service rates that minimize a suitable long-run average cost, subject to an upper bound on the job rejection-rate (quality-of-service constraint). We solve this problem of finding and computing the optimal control under two cases: When the action space is unbounded (i.e. $[0, \infty)$) and when it is bounded (i.e. $[0, \bar{\mu}]$, for some $\bar{\mu} > 0$). We also numerically compute and compare the solutions for different specific choices of the cost function.

CHAPTER 1. OVERVIEW

1.1 Introduction

Most of literature in queuing systems and their optimal characterization is devoted to systems that have holding cost through which the congestion concerns are expressed. The objective in those kind of problems is to balance the trade-off between effort cost function on one hand and the holding cost on the other hand. However, there is another way to include congestion in the problem that fits better in some applications of queuing systems in real-world problems (telecommunication problems, etc.). The alternative way puts a constraint on the queue overflow probability and controls the quality of service (or job-rejection rate from another point of view). We consider a finite-buffer queue with effort cost function, holding cost and the mentioned constraint and provide the solution that minimizes its associated long-run average cost. The arrivals are assumed to happen according to a Poisson process and service rates are exponentially distributed. We provide the solution for two different cases: One with bounded action space and the other with unbounded action space.

Finite-buffer queues are used in modeling static point-to-point wireless link. The long-run average cost is average energy consumption cost (effort cost function) and holding cost. Seeking the optimal service rate for the wireless provider that minimizes the long-run average cost is the objective in this problem.

Following the general idea of constrained Markov decision process (see [2] and [19] for more details) we add a Lagrangian cost to our problem settings rather than a constraint to facilitate solving the problem. We refer to this new problem as *auxiliary problem*. Our Bellman equations are shown to be quite the same for two cases, therefore utilization of the same methodology is possible. A "verification theorem"is proved using the theory of birth-and-death processes (see [15] for details) to help us in characterizing the optimal policy for the auxiliary problem. Based on the optimal solution of auxiliary problem and also the existence of one-to-one mapping between the constraint upper bound and the Lagrangian cost, we are able to find the optimal policy for the *main problem*.

1.2 Literature Review

There is a rich set of literature associated with queuing systems and optimal policy characterization for them. The books of Sennot [11] and [10] study a variety of problems related to dynamic controls of queuing systems and provide background on general methodologies to solve them. The works that are very close to our work are the works that have specifically studied optimal service rate characterization or joint admission and service rate control. We will consider these two types of works separately in following two paragraphs.

As a related work to our problem settings, Stidham and Weber [13] provide a way to prove monotonocity of optimal service rate in one-server queuing models. They used Bellman equations with undiscounted cost structure to prove their observations under the assumption of unboundedness of holding cost vector and boundedness of action space. Under weaker assumptions (less-than-geometric growth for holding cost vector and no bound on action space), George and Harrison [8] provide an elegant method in controlling a Markovian queue system with infinite buffer to minimize the long-run average cost. They use the idea of holding cost truncation to develop an iterative algorithm for finding nearly optimal policy.The motivation behind their study is the work of Wijngaard and Stidham[16] that consider characterizing of optimal policy in a certain type of MDP using Bellman equations and the work of Jo [18] who writes out balance equations for the queue and uses optimization theory in finding the optimal policy for his problem setting. Like majority of other works in queuing theory (see Stidham and Weber [13] and Crabill [6] for instance) balancing the trade-off between holding cost and effort cost function is the key in characterizing the optimal policy in their work. On the other hand, in the work of Ata [3], he studies the trade-off between effort cost function and the constraint same as ours on the overflow probability. His work has applications in control of wireless systems.

Some extensions on [8] is the work of Ata and Shneorson [4] and Adusumilli and Hasenbein [1] who also included admission control in their problem settings.[4] minimized the cost for a problem that the controller can dynamically change both arrival rates and service rates and use that to address a pricing problem. The action space is bounded for both arrival and service rates. Adusumilli and Hasenbein

[1] considered a problem that the controller can simultaneously decide whether or not admit arriving customers and control service rate. The buffer size is not fixed in their settings and they try to find the optimal buffer size. Unlike Guo and Hernández [20] for each state the action space is unbounded in their setting. They also showed that for their Semi-MDP model, one can always stick to a deterministic policy rather than a probabilistic one in seeking the optimal policy. This observation is an extension on the book by Hernández and Lasserre [21] that provides some insights on the same-observation for a discrete-time MDP. Their results are in agreement with Koole [22] who proved monotonicy of optimal policy for the discounted cost version of their problems with bounded action space.

1.3 Thesis Structure

The structure of this thesis is as follows. Chapter 2 provides a complete description of problem in words and mathematically. It also provides the methodology to solve the problem. Additionally, it includes some numerical examples to make the methods clear to reader. Chapter 3 provides a general conclusion to this thesis and also possible directions of further study.

CHAPTER 2. RATE CONTROL OF A QUEUE WITH QUALITY-OF-SERVICE CONSTRAINT UNDER BOUNDED AND UNBOUNDED ACTION SPACES

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Abstract

We consider a simple Markovian queue with Poisson arrivals and exponential service times for jobs. The controller can choose service rates from a specified action space depending on number of jobs in the queue. The queue has a finite buffer and when full, new jobs get rejected. The controller's objective is to choose optimal (state-dependent) service rates that minimize a suitable long-run average cost, subject to an upper bound on the job rejection-rate (quality-of-service constraint). We solve this problem of finding and computing the optimal control under two cases: When the action space is unbounded (i.e. $[0, \infty)$) and when it is bounded (i.e. $[0, \bar{\mu}]$, for some $\bar{\mu} > 0$). We also numerically compute and compare the solutions for different specific choices of the cost function.

2.1 Introduction

In this chapter, we investigate a dynamic optimal control of a single-server queue with finite buffer size N (a positive integer), subject to a constraint on the overflow probability. Arrival of jobs to the

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system happens according to a Poisson process with rate $\lambda > 0$ (to simplify notation, we assume w.l.o.g $\lambda = 1$) and the service time for completing a job follows the exponential distribution. The service rates can be chosen by a "controller" depending on the queue-length (backlog in the system) from a specified action space. If the current queue length is n, the controller chooses a service rate μ_n from the action space. Once the buffer is full, the incoming jobs get rejected (See Fig. 2.1). The upper bound on the buffer overflow rate is $\beta \in [0, 1]$. The long-run average cost function has two components: a service cost or effort cost $c(x)$ when the service rate x is chosen and a holding cost h_n when the queue-length is n. In this chapter, we deal with two types of action space: unbounded action space $A = [0, \infty)$ and when the action space is bounded, i.e. $A = [0, \bar{\mu}]$ for some $\bar{\mu} > 0$. In each case, we solve the dynamic control problem of minimizing the cost subject to the constraint that the buffer overflow probability does not exceed β. This restriction can be viewed as the job-rejection rate constraint or the quality-of-service condition. We solve this control problem by constructing a suitable auxiliary control problem where the constraint appears as the Lagrangian component of the modified cost. Finally, we numerically solve and compare the solution for different choices of the service and holding costs.

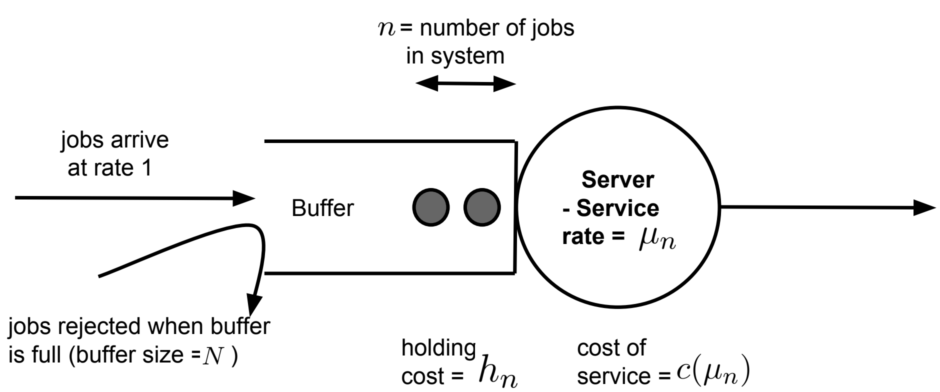


Figure 2.1: Schematic diagram of the queueing model with ajustable service rate

There is a vast body of research on *dynamic control of queuing systems* that uses Markov Decision Process (MDP) framework (see [14], [12], [11], [5] and the references therein). There has been an increase in the study of such networks in the context of quality management of wireless networks that are getting more and more prevalent in the modern applications ([3], [9], [17], [7]). The finite buffer queueing system serves as a model for a static point-to-point wireless link and the long-run average cost is a combination of the average energy consumption per unit time (control cost) as well as customer satisfaction cost (holding cost). The control problem is about finding the optimal rate of service for the wireless provider that minimizes this cost.

This chapter is closely related to the two articles: [3] and [8]. In [8], the authors consider a similar queueing system and similar cost function but with infinite buffer (and subsequently no restriction on overflow probability) and an *unbounded* action space. The article [3] considers similar queueing system with finite buffer size and a constraint on the buffer overflow probability, but with *bounded* action space and no holding cost. Our article attempts to extend both these papers by considering a more general cost function that has both service/effort cost and holding cost as well as a constraint on the buffer overflow rate. We consider a queue with finite buffer size, but address both the cases: *Case 1:* when the action space is unbounded as well as *Case 2:* when the action space is bounded. We use methodology similar to Ata [3] and [8] in the two cases. In both cases, we solve an auxiliary problem where the buffer overflow rate is added to the cost as a Lagrangian component and this solution, in turn, gives us the solution to the original problem with constraint.

Structure of the rest of this chapter is as follows. We describe the model and the main control problem in detail in Section 2.2. We describe the auxiliary Lagrangian problem and solve it in Section 2.3. In section 2.4, we obtain the solution to the main problem using the solution to the auxiliary problem. Finally, in 2.5, we illustrate our methodology numerically using specific choices of the cost functions and model parameters. The Appendix in Section 2.5 discusses the importance of a key convexity assumption on $c(x)$ as well as provides proof of a Lemma.

2.2 Control Problem Formulation: Main Problem

This section provides the precise mathematical formulation for the queuing system. As described in the introduction, we consider a simple queueing model with Poisson process arrivals happening at a rate $\lambda = 1$ (Fig. 2.1). The queue has finite storage capacity N, and new jobs get rejected when the buffer is full. The controller chooses a service rate μ_n from the action space, depending on the number of jobs in the queue $n \leq N$. We assume $\mu_0 = 0$ and assume $\mu_n > 0$ when $n \geq 1$. For each *policy* $\mu = (\mu_1, \ldots, \mu_N)$ that the controller chooses, one gets a unique steady state $p(\mu) = (p_0(\mu), p_1(\mu), ..., p_N(\mu))$, that satisfies:

$$p_n(\mu) = \mu_{n+1} p_{n+1}(\mu) \qquad \text{for} \qquad n = 0, 1, \ldots, N-1 \tag{1}$$

and

$$\sum_{n=0}^{N} p_n(\mu) = 1 \tag{2}$$

Using (1) and (2), one can get explicit form of $p(\mu)$ as follows:

$$p_n(\mu) = p_N(\mu) \prod_{i=n+1}^{N} \mu_i \qquad n = 0, 1, \ldots, N-1 \tag{3}$$

where

$$p_N(\mu) = [1 + \mu_N + \mu_N \mu_{N-1} + \cdots + \mu_N \mu_{N-1} \ldots \mu_1]^{-1}. \tag{4}$$

As mentioned earlier, we consider two cases for the action space set:

$$\begin{aligned} &\textbf{\textit{Case 1:}} \text{ unbounded action space } A = [0, \infty), \\ &\textbf{\textit{Case 2:}} \text{ bounded action space } \bar{A} = [0, \bar{\mu}], \quad \text{for some } \bar{\mu} > 0. \end{aligned} \tag{5}$$

The upper bound on the buffer overflow rate is $\beta \in [0, 1]$. Hence, an admissible policy μ in each of the two cases would be a vector $\mu = (\mu_1, \ldots, \mu_N)$, that satisfies all the conditions above and satisfies $p_N(\mu) \leq \beta$ (note that the rate at which jobs are rejected is modeled here as $p_N(\mu)$ as in [3]). More specifically, the set of admissible policies are:

$$\begin{aligned} &\textit{Case 1:} \quad \Omega = \{\mu \in R^N : \mu_n \in A \backslash \{0\} \text{ for } n = 1, \ldots, N, p_N(\mu) \leq \beta\}, \\ &\textit{Case 2:} \quad \bar{\Omega} = \{\mu \in R^N : \mu_n \in \bar{A} \backslash \{0\} \text{ for } n = 1, \ldots, N, p_N(\mu) \leq \beta\}. \end{aligned} \tag{6}$$

There is a cost component for making jobs wait for service in the system (holding cost) as well as a cost for providing higher service rates (service/effort cost). We model the holding cost incurred when the queue length is n as h_n for $n = 0, 1, \ldots, N$ and assume $0 \leq h_0 \leq h_1 \leq \cdots \leq h_N < \infty$. The service cost is given by $c(x)$ per unit time for providing service rate x. We assume that $c(x)$ is a continuous, differentiable function. Moreover, it is assumed to be increasing and strictly convex in x with $c(0) = 0$. The controller's objective (*in each case*, see (5)) is to minimize, the long-run average cost:

$$z(\mu) = \sum_{n=1}^{N} p_n(\mu) \left[c(\mu_n) + h_n\right] + p_0(\mu) h_0, \tag{7}$$

over all admissible policy μ in Ω (in *Case 1*) or in $\bar{\Omega}$ (in *Case 2*). We also denote the optimal long-run average cost rate for both cases as follows:

$$\textit{Case 1:}\;\; z^* \equiv \inf\{z(\mu) : \mu \in \Omega\}, \qquad \textit{Case 2:}\;\; \bar{z}^* \equiv \inf\{z(\mu) : \mu \in \bar{\Omega}\}. \tag{8}$$

In the rest of the chapter, we describe a method for finding optimal policies in each case, which are defined as a policy $\mu^* \in \Omega$ such that $z(\mu^*) = z^*$ for *Case 1* and a policy $\bar{\mu}^* \in \bar{\Omega}$ such that $z(\bar{\mu}^*) = \bar{z}^*$ for *Case 2*.

Remark 1 *To solve* Case 1 *with unbounded action space, we need to assume (as in [8]) an additional assumption on $c(x)$:*

$$\lim_{w\to\infty} \inf\{c(x)x^{-1} : x \in A, x \geq w\} = \infty \tag{9}$$

We do not need such condition in Case 2. *We will discuss what happens if this condition is not satisfied in* Case 1 *in the discussion in Appendix A at the end of the chapter.*

For the rest of the chapter, we refer to the problem in this section as the "main problem"(as opposed to the "auxiliary problem"described and solved in the next section).

2.3 Auxiliary Problem

For solving the main problem, the theory of dynamic programming is not directly applicable because of the constraint on overflow probability. Hence, as in [3] we will utilize the constrained MDPs (see [2]) to convert our problem to an *auxiliary problem*, which is solvable using dynamic programming techniques. We replace the constraint on the overflow probability by a penalty cost for rejecting jobs (when buffer is full) as a Lagrangian component of the cost function and solve this modified problem by solving the corresponding Bellman equations. In next section, we will use the solution to this modified problem to provide answers to our main problem described earlier.

In the auxiliary problem, the description about the queueing system in Section 2.2 stays the same (in both cases), except for the definition of admissible control vector μ and the form of the cost function as follows. We assume that an admissible control vector $\mu = (\mu_1, \ldots, \mu_N)$ takes values in

$$\begin{aligned} &\textit{Case 1:} \quad \Omega' = \{\mu \in R^N : \mu_n \in A\backslash\{0\} \text{ for } n = 1, \ldots, N\}, \\ &\textit{Case 2:} \quad \bar{\Omega}' = \{\mu \in R^N : \mu_n \in \bar{A}\backslash\{0\} \text{ for } n = 1, \ldots, N\}. \end{aligned} \tag{10}$$

Clearly, any policy μ from above sets is still ergodic and subsequently equations (1) and (2) remain valid. Also, for a fixed rejection penalty $\kappa > 0$, the modified objective function $\gamma(\mu)$ is defined as:

$$\gamma(\mu) = \sum_{n=1}^{N} p_n(\mu)[c(\mu_n) + h_n] + p_0(\mu)h_0 + p_N(\mu)\kappa \tag{11}$$

We also define optimal policies (of the auxiliary problem) and costs, in *each case*, as follows.

$$\begin{aligned} &\textit{Case 1:} \quad \gamma^* \equiv \inf\{\gamma(\mu) : \mu \in \Omega'\}, \ \ \mu^* \equiv \arg\min\{\gamma(\mu) : \mu \in \Omega'\}, \\ &\textit{Case 2:} \quad \bar{\gamma}^* \equiv \inf\{\gamma(\mu) : \mu \in \bar{\Omega}'\}, \ \ \bar{\mu}^* \equiv \arg\min\{\gamma(\mu) : \mu \in \bar{\Omega}'\}. \end{aligned} \tag{12}$$

Before we solve the auxiliary problem, we introduce two key functions for our analysis. In each of the two cases, for $y \geq 0$, define

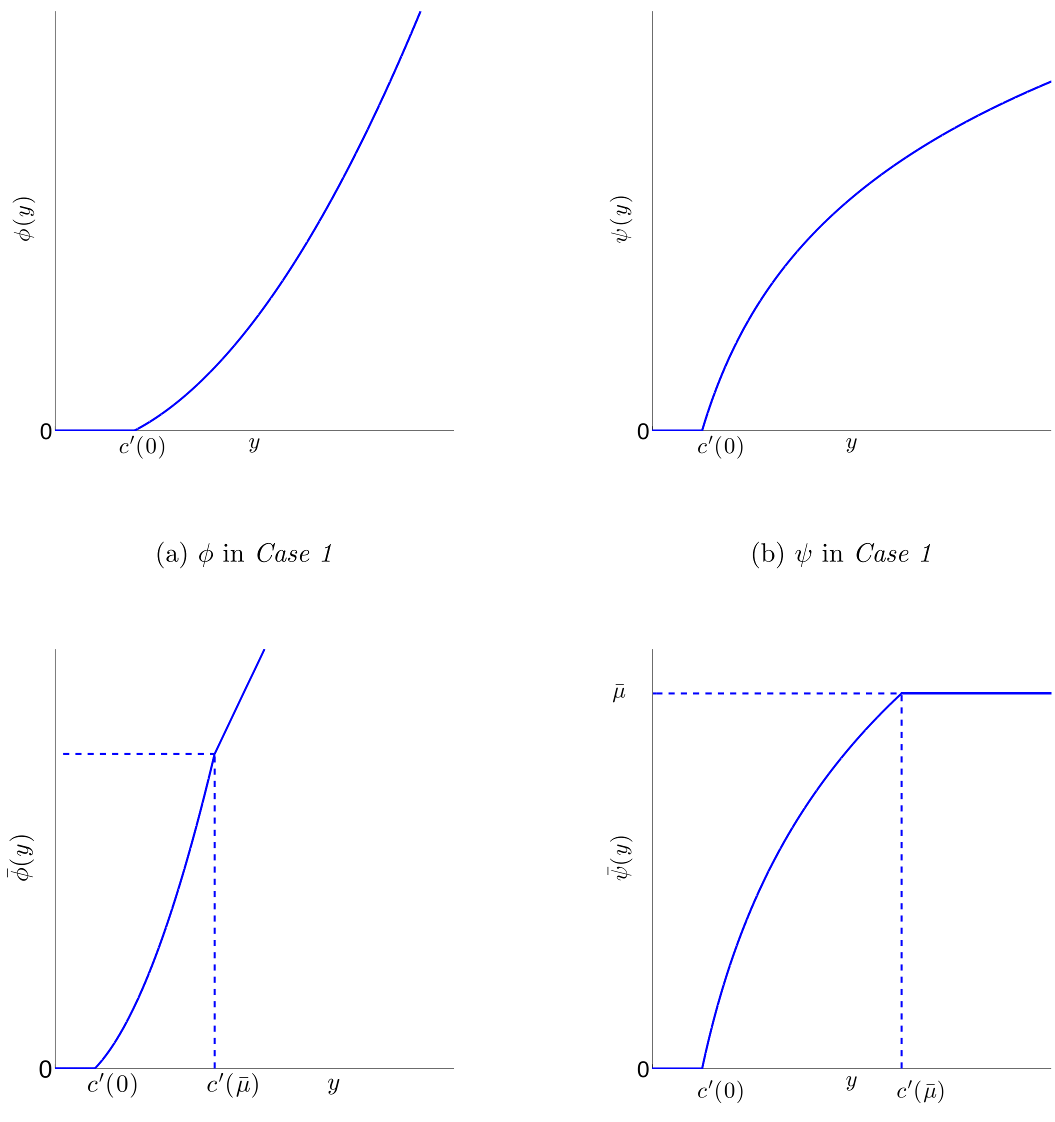
$$\textit{Case 1:} \ \ \phi(y) \equiv \sup_{x \in A}\{xy - c(x)\}, \ \ \psi(y) \equiv \arg\max_{x \in A}\{xy - c(x)\}, \tag{13}$$

$$\textit{Case 2:} \ \ \bar{\phi}(y) \equiv \sup_{x \in \bar{A}}\{xy - c(x)\}, \ \ \bar{\psi}(y) \equiv \arg\max_{x \in \bar{A}}\{xy - c(x)\}. \tag{14}$$

More details on these functions can be found in [8] and [3]. The following key properties of these functions have been proved in these papers and we will be using them without proof for our analysis. Fig. 2.2 gives an example of how these functions behave in the two cases.

Proposition 2 *In* ***Case 1:*** $\phi(.)$ *is continuous, non-decreasing and convex on* $[0, \infty)$ *with* $\phi(y) = 0$ *and* $\lim_{y\to\infty} \phi(y) = \infty$*. Moreover, it is strictly convex and strictly increasing on* $[c'(0), \infty)$*.*
In ***Case 2:*** $\bar{\phi}(.)$ *is continuous, non-decreasing and convex on* $[0, \infty)$ *with* $\bar{\phi}(y) = 0$ *and* $\lim_{y\to\infty} \bar{\phi}(y) = \infty$*. Specifically, it is strictly convex and increasing on* $[c'(0), c'(\bar{\mu})]$ *and is an affine function on* $(c'(\bar{\mu}), \infty)$*.*

In addition, as established in [8] and [3], since $c(.)$ is assumed to be continuous, strictly convex and increasing, we have that ${c'}^{-1}(.)$ is a continuous and increasing function. As a result, the functions



(a) ϕ in *Case 1*

(b) ψ in *Case 1*

(c) $\bar{\phi}$ in *Case 2*

(d) $\bar{\psi}$ in *Case 2*

Figure 2.2: Illustration of ϕ, ψ and the corresponding $\bar{\phi}, \bar{\psi}$ functions in two cases

$\psi(.), \bar{\psi}(.)$ are continuous and non-decreasing on $[0, \infty)$ and can be written explicitly as

$$\psi(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq c'(0) \\ (c')^{-1}(y) & \text{if } y > c'(0) \end{cases} \tag{15}$$

and

$$\bar{\psi}(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq c'(0) \\ (c')^{-1}(y) & \text{if } c'(0) < y < c'(\bar{\mu}) \\ \bar{\mu} & \text{if } y \geq c'(\bar{\mu}) \end{cases} \tag{16}$$

In addition, the following holds.

$$\phi(y) = \int_0^y \psi(u)du, \quad \bar{\phi}(y) = \int_0^y \bar{\psi}(u)du. \tag{17}$$

2.3.1 Solution of the Auxiliary Problem

2.3.1.1 Bellman Equations

This section provides Bellman equations on both cases as a starting step toward computation of optimal policies.

***Case 1*:** In this case, the Bellman equations are similar to those in [8] with appropriate modifications for our problem settings. Using continuous-time MDPs (CTMDPs) theory one can write Bellman equations for relative cost functions denoted by ν_n:

$$\nu_1 = \nu_0 + \gamma - h_0, \tag{18a}$$

$$\nu_n = \min_{x \in A} \left\{ \frac{c(x) + h_n - \gamma + \nu_{n+1} + x\nu_{n-1}}{1 + x} \right\}, \qquad n = 1, \ldots, N-1, \tag{18b}$$

$$\nu_N = \min_{x \in A} \left\{ \frac{c(x) + h_N + \kappa - \gamma}{x} + \nu_{N-1} \right\}. \tag{18c}$$

Here ν_n shows the optimal cost for the shortest path stochastic control problem starting from state n until next time the system is in state m as a reference state. To justify the equations intuitively, note that when the system is in state $n \neq N$ the average time spent on that state is $\frac{1}{1+x}$. Then with probability of $\frac{1}{1+x}$ an arrival occurs before departure and occurrence of departure before arrival has probability of $\frac{x}{1+x}$. Also in state $n = N$ we are sure that no arrival happens as new arrivals will be rejected. The average time the system stays in state N is $\frac{1}{x}$ and with probability of 1, the system moves to state $N - 1$.

Now setting $y_n = \nu_n - \nu_{n-1}$ for $n = 1, \ldots, N$ and using the special functions introduced in (13)-(14) as in [8], the equations (18a)-(18c) reduce to:

$$y_1 = \gamma - h_0, \tag{19a}$$

$$y_{n+1} = \max_{x \in A}\{xy_n - c(x)\} - h_n + \gamma \; = \; \phi(y_n) + \gamma - h_n, \qquad n = 1, \ldots, N-1, \tag{19b}$$

$$\kappa = \max_{x \in A}\{xy_N - c(x)\} - h_N + \gamma \; = \; \phi(y_N) + \gamma - h_N. \tag{19c}$$

Case 2: In this case, since the action space is bounded, it is possible to equalize the average time between transitions. One can use theory of CTMDPs along with uniformization techniques (see [3]) to write the Bellman equations for relative cost functions $\bar{\nu}_n$:

$$\bar{\nu}_1 = \bar{\nu}_0 + \bar{\gamma} - h_0, \tag{20a}$$

$$\bar{\nu}_n = \min_{x \in \bar{A}}\left\{\frac{c(x) + h_n - \bar{\gamma} + x\bar{\nu}_{n-1} + \bar{\nu}_{n+1} + (\bar{\mu} - x)\bar{\nu}_n}{1 + \bar{\mu}}\right\}, \;\; n = 1, \ldots, N-1, \tag{20b}$$

$$\bar{\nu}_N = \min_{x \in \bar{A}}\left\{\frac{c(x) + h_N + \kappa - \bar{\gamma} + x\bar{\nu}_{n-1} + (\bar{\mu} - x)\bar{\nu}_N}{\bar{\mu}}\right\}. \tag{20c}$$

In fact we are using uniformization to convert our CTMDP to a discrete-time MDP (DTMDP) by accelerating the transition rates that are small on average; therefore, the transition rate is $1 + \bar{\mu}$ in all states. Having this done, we need to take into account possibility of staying at same state after transition (see [5] for more details).

Defining $\bar{y}_n$ as $\bar{y}_n = \bar{\nu}_n - \bar{\nu}_{n-1}$ and using the special functions introduced in (13)-(14) as in [3], we can rewrite (20a)-(20c) as

$$\bar{y}_1 = \bar{\gamma} - h_0, \tag{21a}$$

$$\bar{y}_{n+1} = \max_{x \in \bar{A}}\{x\bar{y}_n - c(x)\} - h_n + \bar{\gamma} \; = \; \bar{\phi}(\bar{y}_n) + \bar{\gamma} - h_n, \qquad n = 1, \ldots, N-1, \tag{21b}$$

$$\kappa = \max_{x \in \bar{A}}\{x\bar{y}_N - c(x)\} - h_N + \bar{\gamma} \; = \; \bar{\phi}(\bar{y}_N) + \bar{\gamma} - h_N. \tag{21c}$$

Note that the equations (19a)-(19c) and (21a)-(21c) are quite similar; therefore, we can exploit similar strategy to characterize the optimal policy in both cases.

2.3.1.2 The Verification Theorem

This section states and proves the "verification theorem", which helps us in characterizing the optimal policies through solutions of the Bellman equations in each case.

Theorem 3 *In **Case 1:** If there exists a $\gamma < \infty$ and vector $y \in \mathbb{R}_+^N$ such that the equations (19a)-(19c) are satisfied, then for any arbitrary ergodic policy μ we have*

$$\gamma \leq \gamma(\mu) \tag{22}$$

and subsequently $\gamma \leq \gamma^*$. Let $\mu_n^* = \psi(y_n)$ for $n = 1, \ldots, N$, where ψ is as defined in (13). If the policy μ^* is ergodic, then it is an optimal policy, i.e. $\gamma(\mu^*) = \gamma^* = \gamma$.
*In **Case 2:** If there exists a $\bar{\gamma} < \infty$ and vector $\bar{y} \in \mathbb{R}_+^N$ such that the equations (21a)-(21c) are satisfied, then for any arbitrary ergodic policy μ we have*

$$\bar{\gamma} \leq \gamma(\mu) \tag{23}$$

and subsequently $\bar{\gamma} \leq \bar{\gamma}^*$. Let $\bar{\mu}_n^* = \bar{\psi}(\bar{y}_n)$ for $n = 1, \ldots, N$, where $\bar{\psi}$ is as defined in (14). If the policy $\bar{\mu}^*$ is ergodic, then it is an optimal policy, i.e. $\gamma(\bar{\mu}^*) = \bar{\gamma}^* = \bar{\gamma}$.

Proof. We only prove the theorem for *Case 1* as the proof for the other case is nearly identical. Using (13) one can write

$$xy_n - c(x) \leq \phi(y_n) \qquad \text{for } x \geq 0 \text{ and } n = 1, \ldots, N \tag{24}$$

Since (24) is valid for all $x \geq 0$ we replace it with μ_n, the nth component of arbitrary ergodic policy μ. Then using (19b) and (19c) one concludes that

$$\mu_n y_n - c(\mu_n) \leq y_{n+1} + h_n - \gamma \qquad \text{for } n = 1, \ldots, N-1 \tag{25}$$

$$\mu_N y_N - c(\mu_N) \leq \kappa + h_N - \gamma \tag{26}$$

Note that the policy μ is ergodic; therefore, one can always find a $p(\mu)$ satisfying (1) and (2). We are allowed to multiply both sides of (25) by $p_n(\mu)$ and then sum over all equations. Then, by (1) and after simplification and rearranging we have

$$\gamma \sum_{n=1}^{N-1} p_n(\mu) \quad \leq \quad \sum_{n=1}^{N-1} p_n(\mu)[c(\mu_n) + hn] + p_N(\mu)\mu_N y_N - p_0(\mu)y_1 \tag{27}$$

We also multiply both sides of (26) by $p_N(\mu)$, thus

$$p_N(\mu)\gamma \quad \leq \quad p_N(\mu)[c(\mu_N) + h_N] + p_N(\mu)\kappa - p_N(\mu)\mu_N y_N \tag{28}$$

Finally, one arrives at the following by adding up (28) to (27) and exploiting equations (19a) and (2)

$$\gamma \quad \leq \quad \sum_{n=1}^{N} p_n(\mu)[c(\mu_n) + h_n] + p_0(\mu)h_0 + p_N(\mu)\kappa \tag{29}$$

The right-hand side of (29) is the long-run average cost rate of our arbitrary policy μ.

To prove optimality of μ^*, one can repeat all these steps by replacing x with μ_n^* so that (24) holds with equality. Having substituted all inequalities with equalities and going through all steps again the optimality of policy μ^* is established. □

2.3.1.3 Policy Characterization

This section suggests the methodology to find the optimal policy for the auxiliary problem.

Case 1: Following our simplified Bellman equations in (19a)-(19c), we can define y_n as a function of γ for $n = 1, \ldots, N$; therefore, for each $\gamma > c'(0) + h_0$ we define

$$y_1(\gamma) = \gamma - h_0 \qquad \text{and} \qquad y_{n+1}(\gamma) = \phi(y_n(\gamma)) + \gamma - h_n \qquad \text{for } n = 1, \ldots, N-1. \tag{30}$$

Also the rejection penalty κ can be defined as a function of γ, thus

$$g(\gamma) = \phi(y_N(\gamma)) + \gamma - h_N \tag{31}$$

Let γ_{N+1} be the value of γ such that $g(\gamma) = y_N(\gamma)$. We also define $\alpha = g(\gamma_{N+1})$. Following the arguments in [8], one can always find a unique γ_{N+1} when the action space is unbounded. As a result,

γ_{N+1} and α will be treated as known values hereinafter. It is also showed in the paper that for each $\gamma > \gamma_{N+1}$, one has

$$0 \leq c'(0) < y_1(\gamma) \leq y_2(\gamma) \leq \cdots \leq y_N(\gamma) \tag{32}$$

Note that $\gamma > \gamma_{N+1}$ guarantees $\gamma > c'(0) + h_0$. The following result is a direct consequence of (32) and the monotonicity of ψ function (see the discussion above (15)).

Proposition 4 *The following statement holds for each γ on (γ_{N+1}, ∞),*

$$0 < \psi(y_1(\gamma)) \leq \psi(y_2(\gamma)) \leq \cdots \leq \psi(y_N(\gamma)). \tag{33}$$

Above result shows that for each $\gamma > \gamma_{N+1}$, the policy $\mu^* = (\psi(y_1(\gamma), \ldots, \psi(y_N(\gamma)))$ derived from equations (30) is admissible for *Case 1.* We still need to prove that for each $\kappa > \alpha$ we can find a unique value of $\tilde{\gamma} > \gamma_{N+1}$ that results in $g(\tilde{\gamma}) = \kappa$ in (31). In other words, there is a one-to-one mapping between κ and γ. Having found this unique $\tilde{\gamma}$, then $\tilde{\gamma}$ and $\tilde{y} = (y_1(\tilde{\gamma}), \ldots, y_N(\tilde{\gamma}))$ satisfy (30) and (31). Lemma 5 is needed to prove this observation.

Lemma 5 *Function $g(.)$ is continuous, convex and strictly increasing for any $\gamma > \gamma_{N+1}$. In addition, we have $\lim_{\gamma \to \gamma_{N+1}} g(\gamma) = \alpha$ and $\lim_{\gamma \to \infty} g(\gamma) = \infty$. Moreover, the function $g^{-1}(.)$ is continuous and strictly increasing on (α, ∞). In addition, we have $\lim_{\kappa \to \alpha} g^{-1}(\kappa) = \gamma_{N+1}$ and $\lim_{\kappa \to \infty} g^{-1}(\kappa) = \infty$*

The proof of the first part of the lemma is obvious: It follows from (31) and Proposition 2. The second part is a result of the relation between a function and its inverse.

As we have established a one-to-one mapping between γ and κ, we define for any $\kappa > \alpha$

$$\gamma(\kappa) = g^{-1}(\kappa) \tag{34}$$

to emphasize dependency of γ on κ. We also define policy $\mu^*(\kappa) \equiv (\mu_1^*(\kappa), \ldots, \mu_N^*(\kappa))$ as

$$\mu_n^*(\kappa) = \psi(y_n(\gamma(\kappa))) \qquad \text{for } n = 1, \ldots, N \tag{35}$$

It is clear that $\gamma(\kappa)$ and vector $(y_1(\gamma(\kappa)), \ldots, y_N(\gamma(\kappa)))$ are joint solutions of (19a)-(19c). As we assumed $\kappa > \alpha$ then (34) guarantees that $\gamma(\kappa) > \gamma_{N+1}$; therefore (33) shows that $\mu^*(\kappa)$ is admissible and also monotone (by Proposition 4 and Lemma 5). Hence, by Theorem 3, optimality of $\mu^*(\kappa)$ is confirmed. This is stated below as the theorem for *Case 1* of the auxiliary problem.

Theorem 6 *The policy $\mu^*(\kappa)$ described in (35) characterizes the optimal policy for each $\kappa > \alpha$. Moreover, the policy $\mu^*(\kappa)$ is monotonic in the queue length, i.e.*

$$0 < \mu_1^*(\kappa) \leq \mu_2^*(\kappa) \leq \cdots \leq \mu_N^*(\kappa),$$

and we have $\gamma_{\mu^*(\kappa)} = \gamma^*(\kappa) = \gamma^*$.

Case 2: As we did in the Case of *Case 1* above, we define vector $\bar{y}$ as a function of γ for $\gamma > c'(0) + h_0$. Thus

$$\bar{y}_1(\gamma) = \gamma - h_0 \qquad \text{and} \qquad \bar{y}_{n+1}(\gamma) = \bar{\phi}(\bar{y}_n(\gamma)) + \gamma - h_n \qquad \text{for } n = 1, \ldots, N-1. \tag{36}$$

Also the rejection penalty κ can be defined as a function of γ, thus

$$\bar{g}(\gamma) = \bar{\phi}(\bar{y}_N(\gamma)) + \gamma - h_N \tag{37}$$

Let $\bar{\gamma}_{N+1}$ be the value of γ that gives us $\bar{g}(\gamma) = \bar{y}_N(\gamma)$. The following Lemma is proved in Appendix B.

Lemma 7 *The value of $\bar{\gamma}_{N+1}$ is unique and for each $\gamma > \bar{\gamma}_{N+1}$ we have*

$$0 \leq c'(0) < \bar{y}_1(\gamma) \leq \bar{y}_2(\gamma) \leq \cdots \leq \bar{y}_N(\gamma). \tag{38}$$

The following Proposition follows immediately from (16) and Lemma 7.

Proposition 8 *Each $\gamma > \bar{\gamma}_{N+1}$ will result in*

$$0 < \bar{\psi}(\bar{y}_1(\gamma)) \leq \bar{\psi}(\bar{y}_2(\gamma)) \leq \cdots \leq \bar{\psi}(\bar{y}_N(\gamma)) \leq \bar{\mu} \tag{39}$$

Therefore the admissibility of policy $\bar{\mu}^* = (\bar{\psi}(\bar{y}_1(\gamma)), \ldots, \bar{\psi}(\bar{y}_N(\gamma)))$ obtained from (36) is established. Defining $\bar{\alpha} = \bar{g}(\bar{\gamma}_{N+1})$, again we validate the existence of a unique value of $\tilde{\gamma}$ for which we have $\bar{g}(\tilde{\gamma}) = \kappa$ for each $\kappa > \bar{\alpha}$. Lemma 9 is stated below without proof as it is similar to Lemma 5.

Lemma 9 *Function $\bar{g}(.)$ is continuous, convex and strictly increasing for any $\gamma > \bar{\gamma}_{N+1}$. In addition, we have $\lim_{\gamma\to\bar{\gamma}_{N+1}} \bar{g}(\gamma) = \bar{\alpha}$ and $\lim_{\gamma\to\infty} \bar{g}(\gamma) = \infty$. Moreover, the function $\bar{g}^{-1}(.)$ is continuous and strictly increasing for any $(\bar{\alpha}, \infty)$. In addition, we have $\lim_{\kappa\to\bar{\alpha}} \bar{g}^{-1}(\kappa) = \bar{\gamma}_{N+1}$ and $\lim_{\kappa\to\infty} \bar{g}^{-1}(\kappa) = \infty$*

Now define for any $\kappa > \bar{\alpha}$

$$\bar{\gamma}(\kappa) = \bar{g}^{-1}(\kappa) \tag{40}$$

and $\bar{\mu}^*(\kappa) \equiv (\bar{\mu}^*_1(\kappa), \ldots, \bar{\mu}^*_N(\kappa))$ as

$$\bar{\mu}^*_n(\kappa) = \bar{\psi}(\bar{y}_n(\bar{\gamma}(\kappa))) \qquad \text{for } n = 1, \ldots, N. \tag{41}$$

The following result states the optimality of the control $\bar{\mu}^*(\kappa)$ in *Case 2* and has proof similar to that of Theorem 6.

Theorem 10 *The policy $\bar{\mu}^*(\kappa)$ defined in (41) characterizes the optimal policy for each $\kappa > \bar{\alpha}$. Moreover, the policy $\bar{\mu}^*(\kappa)$ is monotonic in the queue length, i.e.*

$$0 < \bar{\mu}^*_1(\kappa) \le \bar{\mu}^*_2(\kappa) \le \cdots \le \bar{\mu}^*_N(\kappa) \le \bar{\mu},$$

and we have $\gamma_{\bar{\mu}^*(\kappa)} = \bar{\gamma}^*(\kappa) = \bar{\gamma}^*$.

2.4 Solution to the Main Problem

In this section, we use the solution to the auxiliary problem described in Section 2.3 to determine the optimal policy for the main problem. A key step that connects the two problems is the long-run overflow probability function $\beta(\kappa)$ in *Case 1* (and $\bar{\beta}(\kappa)$ in *Case 2*) under the optimal policy of the auxiliary problem $\mu^*(\kappa)$ in *Case 1* (and $\bar{\mu}^*(\kappa)$ in *Case 2*).

Case 1: Using (4) and (35) one can write

$$\beta(\kappa) = [1 + \mu^*_N(\kappa) + \mu^*_N(\kappa)\mu^*_{N-1}(\kappa) + \ldots + \mu^*_N(\kappa)\mu^*_{N-1}(\kappa)\ldots\mu^*_1(\kappa)]^{-1}, \qquad \kappa > \alpha. \tag{42}$$

The following proposition describes the behavior of this function $\beta(\kappa)$.

Proposition 10 *$\beta(.)$ is continuous and strictly decreasing on (α, ∞). In addition, we have $\lim_{\kappa\to\infty} \beta(\kappa) = 0$ and there exists a unique $\kappa^* \in (\alpha, \infty)$ such that $\beta(\kappa^*) = \beta$ for $\beta \in (0, b)$, where $b = \beta(\alpha)$.*

Proof. Continuity of $\psi(.)$ function on $[0, \infty)$ along with the fact that $y_n(.)$ is continuous and its image is a subset of $(c'(0), \infty)$ for $\gamma > \gamma_{N+1}$ (c.f. (32)) result in continuity of $\mu_n^*(\kappa)$ on (α, ∞) for $n = 1, \ldots, N$. Subsequently, continuity of $\beta(\kappa)$ on (α, ∞) is confirmed using (42).

To confirm that $\beta(.)$ is strictly decreasing over (α, ∞), one notes that $\psi(.)$ and $y_n(.)$ are strictly increasing and it concludes that $\mu_n^*(.)$ is strictly increasing over (α, ∞) for $n = 1, \ldots, N$. Using (42) one establishes that $\beta(.)$ is strictly decreasing on (α, ∞).

Eventually, as we increase κ infinitely, $\gamma(\kappa)$ increases infinitely and as the result $y_n(\gamma(p))$ and $\psi(y_n(\gamma(p)))$ go toward infinity for $n = 1, \ldots, N$. Again using (42) one arrives at $\beta(\kappa) \to 0$ as $\kappa \to \infty$. The last part is obvious from the properties of β established in the first part of the proposition. See Fig. 2.3 for an illustration of this function. □

Now we state the main theorem for *Case 1*. It connects our solution of the auxiliary problem to the solution of the main problem in *Case 1*.

Theorem 11 *Given a $\beta \in (0, b)$, existence of a unique $\kappa^* > \alpha$ that gives us $\beta(\kappa^*) = \beta$ is guaranteed and policy $\mu^*(\kappa^*)$ is optimal. In addition, we have $z(\mu^*(\kappa^*)) = z^* = \gamma^* - \beta\kappa^*$.*

Proof. Proposition 10 validates existence and uniqueness of that κ^*. In addition, from Proposition 4 one can conclude that the policy $\mu^*(\kappa^*)$ is admissible as $\kappa > \alpha$. To compare long-run average cost of $\mu^*(\kappa^*)$ with other admissible policies, let policy μ be an admissible policy for the main problem. Then from (29) and after rearranging we have

$$\sum_{n=1}^{N} p_n(\mu)[c(\mu_n) + h_n] + p_0(\mu)h_0 \geq \gamma(\kappa^*) - p_N(\mu)\kappa^* \tag{43}$$

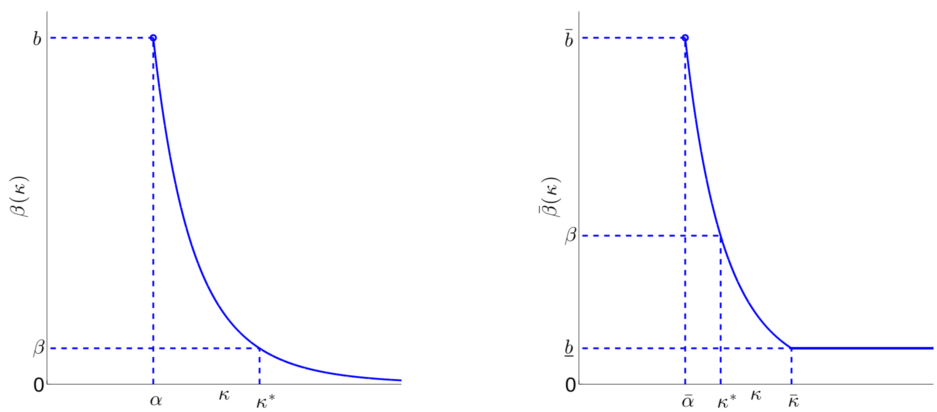
From (43) and admissibility of μ

$$\sum_{n=1}^{N} p_n(\mu)[c(\mu_n) + h_n] + p_0(\mu)h_0 \geq \gamma(\kappa^*) - \beta\kappa^* \tag{44}$$

On the other hand, we have for policy $\mu^*(\kappa^*)$

$$\sum_{n=1}^{N} p_n(\mu^*(\kappa^*))[c(\mu_n(\kappa^*)) + h_n] + p_0(\mu)h_0 = \gamma(\kappa^*) - \beta\kappa^* \tag{45}$$

Clearly, (44) and (45) confirms that the cost associated with any arbitrary admissible policy μ is larger than cost of policy $\mu^*(\kappa^*)$. Hence, by (7) and (11), we conclude that $z(\mu^*(\kappa^*)) = z^* = \gamma^* - \beta\kappa^*$. $\square$



(a) Case 1 (b) Case 2

Figure 2.3: Comparison of illustrative β and $\bar{\beta}$ functions

Case 2: Based on our definition for $\bar{\beta}(\kappa)$, (4) and (41) we have

$$\bar{\beta}(\kappa) = [1 + \bar{\mu}_N^*(\kappa) + \bar{\mu}_N^*(\kappa)\bar{\mu}_{N-1}^*(\kappa) + \cdots + \bar{\mu}_N^*(\kappa)\bar{\mu}_{N-1}^*(\kappa)\ldots\bar{\mu}_1^*(\kappa)]^{-1}, \qquad \kappa > \bar{\alpha}. \tag{46}$$

Before stating the counterpart of Proposition 10 for *Case 2*, we define

$$\underline{b} = \frac{\bar{\mu} - 1}{(\bar{\mu})^{N+1} - 1}, \quad \text{and} \quad \bar{\kappa} = \bar{g}\left(c'(\bar{\mu}) + h_0\right).$$

Proposition 12 *$\bar{\beta}(.)$ is continuous and strictly decreasing on $(\bar{\alpha}, \bar{\kappa})$. In addition, on $[\bar{\kappa}, \infty)$ we have $\bar{\beta}(\kappa) = \underline{b}$ and and there exists a unique $\bar{\kappa}^* \in (\bar{\alpha}, \infty)$ such that $\bar{\beta}(\bar{\kappa}^*) = \beta$ for $\beta \in (\underline{b}, \bar{b})$, where $\bar{b} = \bar{\beta}(\bar{\alpha})$.*

Proof. The proof of continuity is similar to that of $\beta(\kappa)$ in *Case 1*, thus we skip it. To prove that $\bar{\beta}(\kappa) = \underline{b}$ for $\kappa \geq \bar{\kappa}$, we just need to confirm that $\bar{\mu}_1^*(\kappa) = \bar{\mu}$ for $\kappa = \bar{\kappa}$. By (36) we can write $\bar{\mu}_1^*(\bar{\kappa}) = \bar{\psi}(\bar{y}_1(\bar{\gamma}(\bar{\kappa}))) = \bar{\psi}(\bar{\gamma}(\bar{\kappa}) - h_0) = \bar{\psi}(\bar{g}^{-1}(\bar{\kappa}) - h_0) = \bar{\psi}(\bar{g}^{-1}(\bar{g}(c'(\bar{\mu}) + h_0)) - h_0) = \bar{\psi}(c'(\bar{\mu})) = \bar{\mu}$.

Then from Theorem 10 we have $\bar{\mu}_n^*(\kappa) = \bar{\mu}$ for $n = 1, \ldots, N$ and using (46) and some straightforward calculation, validity of our observation is established for $\kappa = \bar{\kappa}$. Finally, monotonocity of $\bar{y}(.)$ and $\bar{\psi}(.)$ generalizes the observation to $\kappa \geq \bar{\kappa}$.

Finally, we confirm that $\bar{\beta}(.)$ is strictly decreasing over $(\bar{\alpha}, \bar{\kappa})$. Using $\bar{\mu}_n^*(\kappa) = \bar{\psi}(\bar{y}_n(\bar{\gamma}(\kappa)))$ along with the fact that $\bar{\gamma}(.)$ and $\bar{y}(.)$ are strictly increasing functions and $\bar{\psi}(.)$ is a non-decreasing one, we conclude that $\bar{\mu}_n^*(.)$ is non-decreasing on $(\bar{\alpha}, \bar{\kappa})$. Exploiting (46) one establishes that $\bar{\beta}(.)$ is non-increasing on $(\bar{\alpha}, \infty)$. Also, since $\bar{\mu}_1^*(\kappa) = \bar{\psi}(\bar{y}_1(\bar{\gamma}(\kappa))) = \bar{\psi}(\bar{\gamma}(\kappa) - h_0)$ and $\bar{\gamma}(.)$ and $\bar{\psi}(.)$ are strictly increasing on $(\bar{\alpha}, \infty)$ and $(c'(0), c'(\bar{\mu}))$ respectively, we can induce that $\bar{\mu}_1^*(.)$ is strictly increasing on $(\bar{\alpha}, \bar{\kappa})$ and that concludes that $\bar{\beta}(.)$ is strictly decreasing on $(\bar{\alpha}, \bar{\kappa})$. The last part is obvious from the properties of β established in the first part of proposition. See Fig. 2.3 for an illustration of this function. $\square$

Finally we state the main theorem for *Case 2*. The proof is omitted as it is similar to that of Theorem 11.

Theorem 13 *Given a $\beta \in (\underline{b}, \bar{b})$, existence of a unique $\bar{\kappa}^* > \bar{\alpha}$ that gives us $\bar{\beta}(\bar{\kappa}^*) = \beta$ is guaranteed and policy $\bar{\mu}^*(\bar{\kappa}^*)$ is optimal. In addition, we have $z(\bar{\mu}^*(\bar{\kappa}^*)) = \bar{z}^* = \bar{\gamma}^* - \beta\bar{\kappa}^*$.*

2.5 Numerical Examples

In this section, we explore some numerical examples to illustrate the solution methodology described in the chapter. All the examples are done for both the cases:

Case 1: $A = [0, \infty)$ and ***Case 2:*** $\bar{A} = [0, \bar{\mu}]$ for fixed $\bar{\mu} > 0$ specified below.

For all the examples, we assume $\lambda = 1$ and $\beta \in [0, 1]$ as before and the holding cost function of the form:

$$h_n = h_0 + s(n - M + 1)^+, n = 0, 1, \ldots, N$$

with fixed real numbers $h_0, s \geq 0$ and positive integers N and M.

Example 1: For our first example, we consider an exponential form for the service cost and these specific values for the parameters:

$$c(x) = e^x - 1 \text{ with buffer size } N = 50,\ \beta = 10^{-20},\ \text{and } h_0 = 0, s = 0.1, M = 1.$$

In addition, we assume $\bar{\mu} = 10$ for *Case 2*. With these choices, we get in *Case 1*:

$$\psi(y) = \begin{cases} 0 & \text{if } 0 \le y \le 1 \\ \ln(y) & \text{if } y > 1 \end{cases} \qquad \text{and} \qquad \phi(y) = \begin{cases} 0 & \text{if } 0 \le y \le 1 \\ y\ln(y) - y + 1 & \text{if } y > 1 \end{cases}$$

and in *Case 2*:

$$\bar{\psi}(y) = \begin{cases} 0 & \text{if } 0 \le y \le 1 \\ \ln(y) & \text{if } 1 < y \le e^{10} \\ 10 & \text{if } y > e^{10} \end{cases} \qquad \text{and} \qquad \bar{\phi}(y) = \begin{cases} 0 & \text{if } 0 \le y \le 1 \\ y\ln(y) - y + 1 & \text{if } 1 < y \le e^{10} \\ 10y + 1 - e^{10} & \text{if } y > e^{10} \end{cases}$$

Fig. 2.4(a) plots the optimal service rate for both cases. Comparing the two graphs, we see that it shows almost the same behavior when the queue length is small - i.e. far from the buffer size $N = 50$. As the queue length increases, *Case 2* applies larger service rates compared to *Case 1* to compensate for its limitation on applying service rates larger than $\bar{\mu} = 10$ until it reaches its maximum service rate at $n = 45$.

Example 2: For second example, we consider a quadratic cost function and the following values for the other parameters:

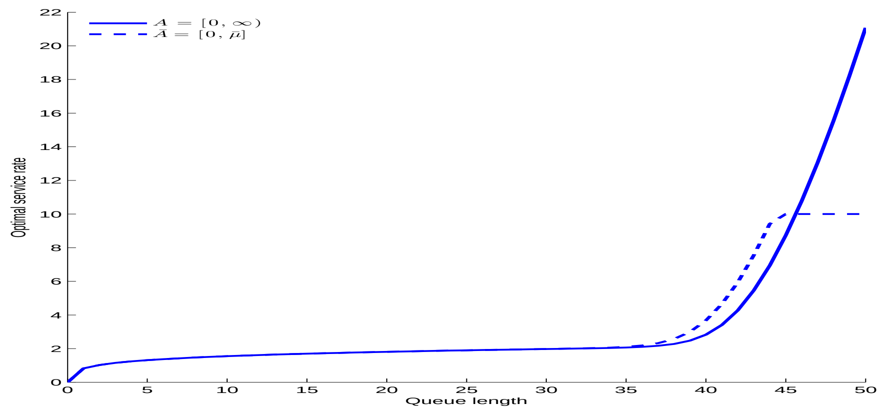
$$c(x) = \frac{1}{2}x^2 \text{ with } N = 10, \beta = 10^{-12},\ \text{and } h_0 = 0, s = 10, M = 1.$$

In addition, we take $\bar{\mu} = 20$ for *Case 2*. Then, in *Case 1* we have for $y \ge 0$

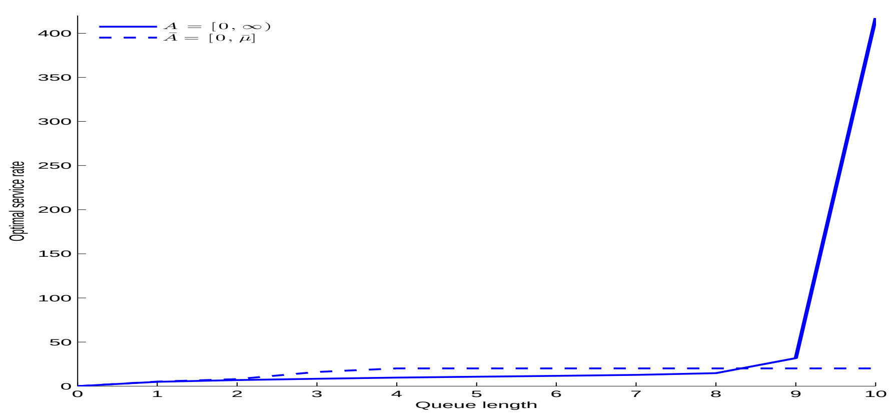
$$\psi(y) = y \qquad \text{and} \qquad \phi(y) = \frac{1}{2}y^2,$$

and for *Case 2*:

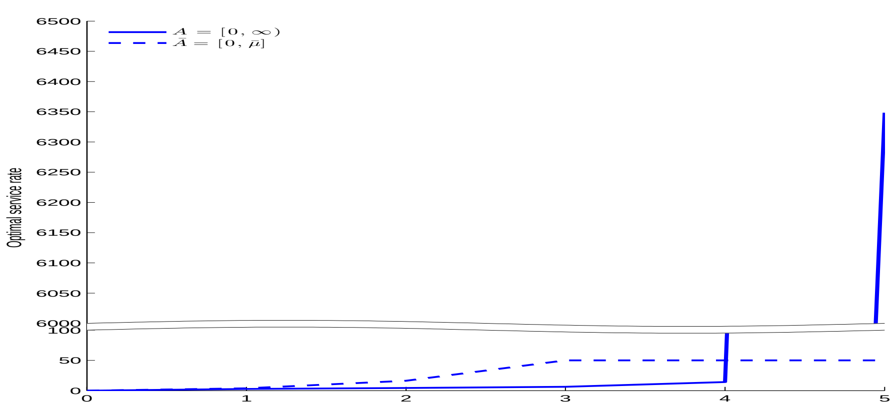
$$\bar{\psi}(y) = \begin{cases} y & \text{if } 0 \le y \le 20 \\ 20 & \text{if } y > 20 \end{cases} \qquad \text{and} \qquad \bar{\phi}(y) = \begin{cases} \frac{1}{2}y^2 & \text{if } 0 \le y \le 20 \\ 20y - 200 & \text{if } y > 20 \end{cases}$$



(a) $c(x) = e^x - 1$.



(b) $c(x) = \frac{1}{2}x^2$.



(c) $c(x) = (x + 1)ln(x + 1)$.

Figure 2.4: Optimal policies in three numerical examples for each of the two cases

Fig. 2.4(b) shows the optimal policy rate in both cases. In this example the difference between κ^* and $\bar{\kappa}^*$ is huge and that makes the optimal rate to reach its maximum very fast (when queue length is around 4) in *Case 2*.

Example 3: Finally for our third example, we consider the following service cost function and parameter values:

$$c(x) = (x+1)\ln(x+1) \text{ with } N = 5, \beta = 10^{-7} \text{ and } h_0 = 0, s = 1, M = 1,$$

and $\bar{\mu} = 50$ for *Case 2*. Then for *Case 1* we have

$$\psi(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq 1 \\ e^{y-1} - 1 & \text{if } y > 1 \end{cases} \quad \text{and} \quad \phi(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq 1 \\ e^{y-1} - y & \text{if } y > 1 \end{cases}$$

and for *Case 2:*

$$\bar{\psi}(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq 1 \\ e^{y-1} - 1, & \text{if } 1 < y \leq \ln(51) + 1 \\ 50 & \text{if } y > \ln(51) + 1 \end{cases} \text{ and } \bar{\phi}(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq 1 \\ e^{y-1} - y & \text{if } 1 < y \leq \ln(51) + 1 \\ 50y - 51\ln(51) & \text{if } y > \ln(51) + 1 \end{cases}$$

Fig. 2.4(c) illustrates optimal policy for third example. Note that we have broken vertical axis to make the figure more clear to reader. The optimal rate in *Case 2* reaches close to the maximum ($\bar{\mu} = 50$) when the queue-length is around 3.

Comparing the three examples, note that the service cost function has gradual slower rate of growth from Example 1 - 3, and hence, the rate of change of the optimal control (as a function of the queue-length) is also smoother in a later example, compared to an earlier one. This effect is more visible when the optimal rate is closer to the upper bound.

Tables 2.1 and 2.2 summarize our results for both cases respectively considering all three cost functions $c(x)$ in the three examples above.

$c(x)$	$e^x - 1$	$\frac{1}{2}x^2$	$(x+1)\ln(x+1)$
β	1e − 20	1e − 12	1e − 7
N	50.00000000000000	10.00000000000000	5.000000000000000
M	1.000000000000000	1.000000000000000	1.000000000000000
s	0.100000000000000	10.00000000000000	1.000000000000000
γ_{N+1}	2.285307016378286	4.890089061547525	2.297603580454379
α	8.958166186885266	14.82824086628696	3.210938206766867
γ^*	2.285307017566225	4.890089305336006	2.303829201039988
κ^*	28.54874985659808	417.3020809625224	9.756135451108200
z^*	2.285307017280737	4.890089304918704	2.303828225426443

Table 2.1: Computational results for case 1

$c(x)$	$e^x - 1$	$\frac{1}{2}x^2$	$(x+1)\ln(x+1)$
β	1e − 20	1e − 12	1e − 7
N	50.00000000000000	10.00000000000000	5.000000000000000
M	1.000000000000000	1.000000000000000	1.000000000000000
s	0.100000000000000	10.00000000000000	1.000000000000000
$\bar{\gamma}_{N+1}$	2.285307016378286	4.890089061547525	2.297603580454379
$\bar{\alpha}$	8.958166186885266	14.82824086628696	3.210938206766867
$\bar{\gamma}^*$	2.285307019577728	5.061191089366425	2.578945446808966
$\bar{\kappa}^*$	101524018148.6422	5825230813.475831	24786.09850823848
$\bar{z}^*$	2.285307018562488	5.055365858552950	2.576466836958142

Table 2.2: Computational results for case 2

Appendices

A About Remark 1

To investigate the result when the assumption in Remark 1 (see (9)) does not hold, let us first consider its failure in *Case 2*. Suppose that we have linear function $c(x) = \theta x$ with $\theta > 0$. Derived results in Sections 2.3 and 2.4 shows that the optimal policy would be $\bar{\mu}_n^* = \bar{\mu}$ for $n = 1, \ldots, N$ given that this policy satisfies the constraint $p_N(\bar{\mu}^*) \leq \beta$; otherwise there is no feasible solution to our problem. Additionally, a simple investigation of the average cost of implementing service rate x which is $\frac{c(x)}{x} = \theta$ justifies this result. While the average cost for implementing any service rate $x \in [0, \bar{\mu}]$ is equal, it clearly makes sense to apply the largest possible service rate $\bar{\mu}$ to offset the imposed congestion costs. Note that although degeneracy can be the case for the auxiliary problem (the situation where $h_N + \kappa < \bar{\gamma}^*$), but it never happens for the main problem as the do-nothing policy is an infeasible solution.

Now we can generalize our result to the *Case 1* by having $\bar{\mu} \to \infty$ in *Case 2.* We have two interpretations for this case; mathematical and practical. Mathematically speaking, one can infer that in this case no optimal policy exists since providing service rate with infinite value always dominates a service rate with finite value. In this case feasibility issues are not our concern as we can always find μ such that $p_N(\mu) \leq \beta$ for any $0 < \beta < 1$. On the other hand, note that here the controller can remove j jobs instantly at any time with the cost of $j\theta$. Instantaneous ejection of the arriving jobs is practically equivalent to give the controller the leverage of rejecting new jobs when the buffer is not full at cost θ. Practically speaking, this is where we need to switch the problem to a new one. A new problem that one should design a queue with an optimal buffer size and optimal service rates while there is a similar-to-our-context constraint on that optimal buffer size. On that problem one can relax the assumption (9) to the following assumption

$$\lim_{w \to \infty} (\inf\{c(x)x^{-1} : x \in A, x \geq w\}) \geq \tilde{\kappa} \tag{47}$$

where $\tilde{\kappa}$ is the rejection cost in that problem. This is where our problem, and those in [8] and [1] overlap and can be investigated separately.

B Proof of Lemma 7

To verify Lemma 7, it is enough to show the existence of a unique $\bar{\gamma}_{N+1}$ be such that $\bar{g}(\gamma) = \bar{y}_N(\gamma)$ and prove that for any $\gamma > \bar{\gamma}_{N+1}$ we have $0 \leq c'(0) < \bar{y}_1(\gamma) \leq \bar{y}_2(\gamma) \leq \cdots \leq \bar{y}_N(\gamma)$. To make it simple, let us consider the problem with an unlimited buffer size. In other words we have the following equations:

$$\bar{y}_1(\gamma) = \bar{\gamma} - h_0, \tag{48a}$$

$$\bar{y}_{n+1}(\gamma) = \bar{\phi}(\bar{y}_n(\gamma)) + \bar{\gamma} - h_n \tag{48b}$$

Furthermore, to facilitate our proof we extend the domain of function $\bar{\phi}$ to $\mathbb{R}$ by defining $\bar{\phi}(y) = 0$ for $y < 0$. That gives us a sequence of $\bar{y}_n(.) : \mathbb{R} \to \mathbb{R}$ functions that are continuous and strictly increasing.

Now, set $\bar{y}_0 = c'(0)$ and define the difference of consecutive terms of vector $\bar{y}$ as

$$\Delta_n(\gamma) = \bar{y}_n(\gamma) - \bar{y}_{n-1}(\gamma) \qquad \text{for } n \geq 1 \text{ and } \gamma \in \mathbb{R} \tag{49}$$

Then immediately we reach the following using (48b).

$$\Delta_{n+1}(\gamma) = \bar{\phi}(\bar{y}_n(\gamma)) - \bar{\phi}(\bar{y}_{n-1}(\gamma)) - (h_n - h_{n-1}) \qquad \text{for } n \geq 1 \text{ and } \gamma \in \mathbb{R} \tag{50}$$

Let γ_n be the root of nth equation, i.e. the solution of $\Delta_n(\gamma) = 0$. We show that a unique sequence of roots $\gamma_1, \gamma_2, \ldots, \gamma_n$ exists that are monotone in n. Moreover, we show that

$$0 \leq c'(0) = \bar{y}_0(\gamma_n) \leq \bar{y}_1(\gamma_n) \leq \bar{y}_2(\gamma_n) \leq \cdots \leq \bar{y}_{n-1}(\gamma_n) = \bar{y}_n(\gamma_n).$$

We argue this using induction method. First note that $\Delta_1(\gamma) = \gamma - h_0 - c'(0)$ and subsequently $\gamma_1 = h_0 + c'(0)$ which is unique. One notes that $\Delta_1(.)$ is continuous and strictly increasing over $[\gamma_1, \infty)$. Next we assume that our observations are true for nth equation and will try to extend them to $(n+1)$th equation. Therefore, we assume a unique γ_n such that $\Delta_n(\gamma_n) = 0$ and $\gamma_1 \leq \cdots \leq \gamma_n$. Furthermore, on $[\gamma_n, \infty)$, $\Delta_n(.)$ is assumed continuous and strictly increasing. The fact that $\bar{\phi}$ is continuous and strictly increasing on $[c'(0), \infty)$ along with our assumption that $\bar{y}_n(\gamma) - \bar{y}_{n-1}(\gamma) \geq 0$ for $\gamma \geq \gamma_n$ lead us to infer that Δ_{n+1} is continuous, strictly increasing and unbounded on $[\gamma_n, \infty)$ from (50). As we have extended our hypothesis to $n+1$ one can infer that for $\gamma \geq \gamma_i$ we have $\bar{y}_i(\gamma) \geq \bar{y}_{i-1}(\gamma)$ and that ends the proof of monotonocity of $\bar{y}_n(.)$ sequence. Moreover, it is immediate from (50) that $\Delta_{n+1}(\gamma_n) = h_{n-1} - h_n \leq 0$. Exploiting that along with continuity, unboundedness and strict monotoncity of $\Delta_{n+1}(.)$ on $[\gamma_n, \infty)$, we establish existence of a unique $\gamma_{n+1} \geq \gamma_n$ such that $\Delta_{n+1}(\gamma_{n+1}) = 0$.

Returning to the problem with limited buffer size N one can immediately see that we can always find a unique γ_{N+1} value such that $\bar{g}(\gamma_{N+1}) = \bar{y}_N(\gamma_{N+1})$ and for each $\gamma > \gamma_{N+1}$ we have $0 \leq c'(0) < \bar{y}_1(\gamma) \leq \bar{y}_2(\gamma) \leq \cdots \leq \bar{y}_N(\gamma)$. The equality situations happen when the holding cost remains the same from a state to its immediate larger state (Note that we assumed holding cost vector to be non-decreasing). □

3. GENERAL CONCLUSION

3.1 Conclusion

We showed that Bellman equations for *Case 1* with unbounded action space and *Case 2* with bounded action spaces are quite similar and that facilitates exploitation of similar strategy in solving the problem.

As the strategy for solving the main problem, we introduced an auxiliary problem whose solution is connected to our main problem solution. We used a Lagrangian cost and replaced it with our constraint on overflow probability. We provided a verification theorem to facilitate solving this auxiliary problem. Additionally, A one-to-one mapping between this Lagrangian cost and the upper bound was shown to exist. Furthermore, we provided the way on how to find this Lagrangian cost value given an upper bound.

Finally, some numerical examples with different parameters and cost functions were provided to make the methodology clear and the results were provided.

3.2 Future Study

During this study, we assumed that arrivals happen according to a Poisson process and service rates are exponentially distributed. This is not always the case in real-world problems and arrivals and service rates can have a general distribution. Relaxing these assumptions and studying the problem in that setting could be a possible future direction.

Moreover, we assumed that regardless of waiting times, the customers do not depart the system without being served (no customer abandonment). Again, this is not always the case and one could investigate this by assuming a general or exponential distribution for the abandonment.

Combination of exponential or general distribution for arrivals, service rates and customer abandonment result in many interesting problems to investigate.

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