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Static and Dynamic Resource Allocation Models for Recovery of Interdependent Systems: Application to the *Deepwater Horizon* Oil Spill

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Abstract Determining where and when to invest resources during and after a disruption can challenge policy makers and homeland security officials. Two decision models, one static and one dynamic, are proposed to determine the optimal resource allocation to facilitate the recovery of impacted industries after a disruption where the objective is to minimize the production losses due to the disruption. The paper presents necessary conditions for optimality for the static model and develops an algorithm that finds every possible solution that satisfies those necessary conditions. A deterministic branch-and-bound algorithm solves the dynamic model and relies on a convex relaxation of the dynamic optimization problem. Both models are applied to the *Deepwater Horizon* oil spill, which adversely impacted several industries in the Gulf region, such as fishing, tourism, real estate, and oil and gas. Results demonstrate the importance of allocating enough resources to stop the oil spill and clean up the oil, which reduces the economic loss across all industries. These models can be applied to different homeland security and disaster response situations to help governments and organizations decide among different resource allocation strategies during and after a disruption.

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1 Introduction

The 2010 explosion on the *Deepwater Horizon* oil rig resulted in the largest marine oil spill in history (Robertson and Krauss 2010). Eleven people died and 16 other employees were injured from the explosion, and nearly 5 million barrels of crude oil spilled into the Gulf of Mexico. The environmental damage, loss of wildlife, and loss of business to several Gulf industries exemplify the far-reaching consequences that disruptions can have on a region, and government policy makers must allocate resources effectively to minimize the impacts of a disruption. Officials who are responsible for helping an economic region recover from such a disruptive event need to understand how the disruption impacts the economy, determine the best allocation resources at different points in time, and analyze how allocating resources to particular industries benefits the regional economy.

This paper—a shorter version of which appears in MacKenzie et al. (2012b)—seeks to help officials resolve those difficulties by developing a resource allocation for regional economic recovery following a disruption. It makes the following contributions to operations in homeland security and disaster management. First, the modeling approach allows a policy maker to determine the level of resources that he or she should allocate to specific industries in order to effectively reduce the adverse impact of a disruptive event. Second, unlike many other homeland security resource allocation models, we focus on post-disruption decision making that seeks to limit the impacts and enhance recovery. Third, because policy makers are required to make decisions over the course of a disruption, we construct a discrete-time dynamic model in addition to a static model. Finally, estimating values for model parameters from a variety of sources enables these models to be applied to the *Deepwater Horizon* oil spill, which adversely impacted several industries in the Gulf region. The application of this analytical approach to the *Deepwater Horizon* oil spill generates insights into where resources should be concentrated if a similar disruption occurs.

This paper provides a theoretical construct that can potentially assist policy makers to allocate resources following a disruption. Although the *Deepwater Horizon* oil spill represents a real-world application of the models, consultation with government officials to estimate parameters would be necessary in order to practically use these models. Section 2 reviews previous optimal resource allocation models and expands on the unique contributions of this paper. Section 3 develops and provides solutions for two decision models: (i) a static model of both direct and indirect impacts from a disruption and (ii) a discrete-time dynamic model where resources are allocated over time. Section 4 applies these models to the *Deepwater Horizon* oil spill and analyzes the sensitivity of model results to key parameters. Concluding remarks appear in Section 5.

2 Literature review

A resource allocation model seeks to answer the fundamental economic question of how to satisfy unlimited wants with limited resources within a specific domain. Resource allocation models are typically formulated as static or dynamic optimization problems with a resource budget serving as a primary constraint. Such resource allocation models have been deployed to analyze several policy-related problems.

Resource allocation models in engineering risk management generally focus on reinforcing different components or building redundancy within a system in order to maximize reliability or minimize failure (Tillman et al. 1970; Misra and Ljubojević 1973; Elms 1997). Guikema and Paté-Cornell (2002) develop an optimization problem in which several components can be upgraded, and the alternatives are whether or not to select a component for upgrading and how much money to spend on upgrading the component. A two-period model (Dillon et al. 2003) examines a problem for NASA where a decision maker minimizes the technical risk of an exploration spacecraft in the first period and allocates the remainder of the budget to minimize the risk of failure during the development phase. A dynamic model (Dillon et al. 2005) extends this two-period model by allowing the decision maker to allocate resources at different points in time to improve reliability.

Homeland security officials have struggled with how to allocate resources to different geographic areas based on risk or cost effectiveness. An analysis of the Department of Homeland Security's fiscal year 2004 budget reveals that the department's allocation to urban areas to protect those areas from terrorism significantly differs from an allocation based on the risk of a terrorist attack (Willis 2007). As Willis (2007) acknowledges, the decision should be based on where the money reduces risk the most rather than on which areas carry the most risk, but data do not exist to estimate the functional relationship between investing in protection and risk reduction. This latter point applies to protection against natural disasters as well. Other resource models for homeland security deploy game theory to help government officials understand how resources should be allocated differently to protect against a strategic actor such as a terrorist (Major 2002; Bier 2007; Zhuang and Bier 2007; Bakir 2011; Shan and Zhuang 2013a,b). Haimes et al. (2008) offer several systems engineering principles to help policy makers balance protective and resilience activities for critical infrastructure protection against both terrorism and natural disasters.

Dynamic resource allocation models seek to efficiently allocate resources at different points in time. Rather than allocating money, many of these models allocate discrete resources such as machines at a work center (Miller and Davis 1978), specific resources to complete a job (Daniels et al. 1997), and operators and cranes to unload or load ships (Dell'Olmo and Lulli 2004). Consequently, these models are formulated as integer programs, whose solutions may require heuristic algorithms for problems of realistic size. The medical field has been a natural application for dynamic resource allocation problems as policy makers seek to understand the best intervention strategies to stop the spread of a disease (Zaric and Brandeau 2000; Brandeau 2005). Dynamic resource allocation models seem to be sparser in the field of disaster management or disaster response although Fiedrich et al. (2000) build a model to determine the placement of machines and equipment to minimize fatalities after an earthquake and Petrovic et al. (2012) provide a model for responding to a wildfire.

Two dynamic allocation models specific to oil spills (Psaraftis and Ziogas 1985; Srinivasa and Wilhelm 1997) focus on tactical decisions to determine the type of equipment to clean up a spill. The decision variables in both models are discrete integers, and the models in this paper focus on strategic decision making as opposed to tactical decision making. A game theoretic model related to oil spills determines the level of safety effort that results from the strategic interaction between an oil company and the government (Hausken and Zhuang 2013). Cheung and Zhuang (2012) analyze the impact of competition between two oil companies on whether a company will follow safety regulations and whether a government will enforce those regulations.

The modeling approach in this paper borrows from the different resource allocation models but also develops new insights and methods to aid policy makers. This paper focuses on post-disruption decision making in order to limit the impacts and enhance recovery. A risk-based input-output model has been proposed to improve preparedness decision making in the face of potential disruptions (Crowther 2008), and such a model used in response decision making quantifies the decision maker's objective by translating direct impacts from a disruption into total production losses in a region. The functional form that maps allocated resources to a reduction in direct impacts is borrowed from the engineering risk analysis literature and is used to identify those industries where resources can most reduce the impacts from a disruption.

The discrete-time dynamic model in this paper allows the decision maker to allocate resources at different points in time. Unlike many other dynamic resource allocation models, our model assumes resources are infinitely divisible like money, as opposed to discrete resources, and the solution to this model relies on a branch-and-bound algorithm for continuous variables. The resource constraint is one constraint for the entire time period as opposed to a budget constraint for each individual period of time, as many dynamic models propose.

Finally, a policy maker can use the models to understand the effect of different factors that may influence the optimal allocation of resources. The models seek to illuminate the relationship between the optimal allocation and parameters such as direct impacts, the effectiveness of allocating resources, the resource budget, and time. By comparing the benefits of allocating resources to help multiple industries simultaneously with the benefits of targeting individual industries, this modeling approach can offer guidance on general recovery efforts versus specific recovery tasks for an industry.

3 Resource allocation models

3.1 Inoperability Input-Output Model

Two resource allocation models, one static and one dynamic, measure the economic consequences from a disruption, and a policy maker wishes to allocate resources to minimize total production loss caused by the disruption. Production losses derive from both direct and indirect impacts. Direct impacts represent production losses that result directly from final consumers reducing their demand or from facilities that are inoperable due to the disruption. Indirect impacts are production losses incurred by industries or firms who depend on those directly impacted industries. Large-scale disruptions such as the 2011 Japanese earthquake and tsunami can induce indirect impacts on a global scale (MacKenzie et al. 2012c).

Industries and economic sectors suffer from indirect impacts because of the interdependencies among industries and infrastructure systems. Two entities are interdependent if each impacts or influences the performance or functionality of the other entity (Rinaldi et al. 2001). Interdependence plays an important role in risk and policy analysis because a disruptive event that directly impacts infrastructure, business, or the economy can induce partial or even total failure in other systems, markets, and businesses that are not directly impacted by the event. A variety of models (see Pederson et al. 2006; Medal et al. 2011) have been proposed to understand and analyze the linkages among critical infrastructure systems. Network models can quantify the vulnerability, resilience, and interdependence of infrastructure systems (Dueñas-Osorio et al. 2007; Johansson and Hassel 2010). System dynamics models attempt to capture the interdependence between infrastructure and humans during a disruption (Conrad et al. 2006).

The models presented in this paper measure the economic interdependence among industries using an input-output model. Modeling economic interdependence in the midst of a disruption seeks to quantify production and demand changes resulting from the disruptive event (Okuyama and Chang 2004). The Leontief (1936) input-output model measures these interdependent impacts by assuming that production changes are demand driven. If a disruption forces an industry to produce less or causes final demand to drop, less product is demanded of suppliers. Linear dependencies among industries are driven completely by these changes in final or intermediate demand. Despite the linear and demand-driven assumptions, input-output models provide a reasonable estimate of the economic impacts of disruptions (Boisvert 1992; Gordon et al. 2005; Okuyama 2008), and the models are supported by a large data collection effort undertaken by governments around the world.

The Inoperability Input-Output Model (IIM) (Santos and Haimes 2004) is a risk-based extension of the Leontief input-output framework. A disruption directly impacts m industries in an economy with a total of n industries, where $m \leq n$. A vector $\mathbf{c}^*$ is of length m, and c_i^* measures the direct impacts, in proportional terms, to industry i. $\mathbf{D} \equiv (\mathbf{I} - \mathbf{A}^*)^{-1}$ is a square matrix of order n, where $\mathbf{A}^*$ is the normalized interdependency matrix in the IIM. $\mathbf{D}$ translates direct impacts into direct and indirect impacts. Each element in the matrix, d_{ji}, calculates the proportional loss in production for industry j due to a loss in production in the directly impacted industry i. Each element on the diagonal of $\mathbf{D}$ is greater than or equal to one because direct impacts in industry i lead to total impacts at least as large as the direct impacts in that industry. If no interdependencies are present, $\mathbf{D}$ is the identity matrix. Because the disruption does not directly impact all n industries but only m industries, we use $\tilde{\mathbf{D}}$, a $n \times m$ matrix whose columns correspond to the directly impacted industries from $\mathbf{D}$. Thus, $\tilde{\mathbf{D}}$ translates the direct impacts in the m industries into direct and indirect impacts for all n industries in the economy. Table 1 depicts $\tilde{\mathbf{D}}$ for the *Deepwater Horizon* application. Most of the off-diagonal elements are on the order of 10^{-2} or 10^{-3}, which demonstrate that interdependencies between any two industries are fairly small. However, the column sums of $\tilde{\mathbf{D}}$ for the Real Estate, Accommodations, and Oil and Gas industries are 2.75, 1.39, and 1.73, respectively. Indirect impacts due to economic interdependencies are 175%, 39%, and 73% of the direct impacts for each of those industries.

Translating the proportional impacts to production losses requires multiplying the total impacts by $\mathbf{x}$, which is a vector of length n representing as-planned production for each industry in the economy. Total production losses due to the disruption is given by $\mathbf{x}^{\intercal}\tilde{\mathbf{D}}\mathbf{c}^*$. The IIM has been used to study a number of disruptions that concern policy makers including terrorist attacks (Haimes et al. 2005), cyber security (Dynes et al. 2007), workforce disruptions (Barker and Santos 2010), and waterway port closures (MacKenzie et al. 2012a).

3.2 Model 1: Static allocation

For the first model, a policy maker wishes to allocate resources to minimize the total production loss caused by the disruption. Model (1) presents the policy maker's problem through optimization. The total budget, Z, is divided into resources allocated to each directly impacted industry, $z_1, \ldots, z_m$, and all industries simultaneously, z_0. These z_i (where $i = 1, \ldots, m$) and z_0, which serve as the decision variables in the optimization problem, are investments to promote recovery following a disruptive event. Under this model, the policy maker's goal is to minimize total production losses in a region as determined by the IIM.

$$\begin{aligned} \text{minimize} \quad & \mathbf{x}^\mathsf{T}\tilde{\mathbf{D}}\mathbf{c}^* \\ \text{subject to} \quad & c_i^* = \hat{c}_i^* \exp\left(-k_i z_i - k_0 z_0^2\right) \qquad i = 1,\ldots,m \\ & z_0 + \sum_{i=1}^{m} z_i \le Z \\ & z_0, z_i \ge 0 \qquad\qquad\qquad\qquad i = 1,\ldots,m \end{aligned} \tag{1}$$

The direct impacts to each industry, c_i^*, is a function of the allocation amounts, k_i (the effectiveness of allocating resources to industry i), k_0 (the effectiveness of allocating resources to all industries simultaneously), and $\hat{c}_i^*$ (direct impacts if no resources are allocated). Direct impacts on an industry can be assessed by (i) estimating the number of consumers that would stop purchasing from an industry because of a disruption or (ii) measuring the amount of production that would be lost if a facility were suddenly closed.

The model assumes that allocating resources reduces the impacts exponentially, which is a frequent assumption in engineering risk problems (Bier and Abhichandani 2003; Guikema and Paté-Cornell 2002; Dillon et al. 2005). As more resources are allocated to an industry, the impacts on an industry decline at a constantly decreasing rate, and investing an additional dollar to reduce risk returns less benefit than investing the first dollar. For each directly impacted industry, the exponential function requires estimating a cost-effectiveness parameter, k_i. As (2) shows, this parameter can be assessed if z_i, the amount of resources needed to reduce the direct impacts on industry i by a given fraction $c_i^* \big/ \hat{c}_i^*$, is known or can be estimated, since

$$k_i = -\frac{\log\left(c_i^*/\hat{c}_i^*\right)}{z_i} \qquad i = 1,\ldots,m. \tag{2}$$

The value of k_i is always greater than or equal to 0 but has no upper bound. We expect k_i to be extremely small for large-scale disruptions where millions of dollars are necessary to reduce the impacts. For example, if it takes \$1 million to reduce the direct impacts by half, $k_i = -\log(0.5)\big/10^6 = 6.9 * 10^{-7} = 0.69$ per \$1 million.

In addition to allocating resource to benefit a single industry, a policy maker can also allocate resources to simultaneously benefit all industries, as represented by the parameter z_0. These resources could include activities such as cleaning the area and removing debris after the disruption, repairing infrastructure that all industries require (e.g., electric power, transportation), and engaging in risk communication efforts to inform the public that a region is safe. The model squares this allocation amount because of an assumption that if a major disruption occurs, allocating resources for these types of activities will not enhance recovery unless a significant amount of resources is allocated. Mathematically, $k_0 < 1$ and squaring z_0 reduces the impact of allocating z_0 if $\sqrt{k_0}z_0 < 1$. Squaring z_0 also assumes that if a substantial amount of resources are allocated to all industries (i.e., if $\sqrt{k_0}z_0 > 1$), the impact of this allocation is enhanced. We base this assumption on a belief that actions such as containing a disruption like an oil spill and rebuilding infrastructure are crucial in order to generate economic activity in a region after a disruption.

Equations (3) - (5) depict the Karush-Kuhn-Tucker (KKT) conditions for optimality, where λ, λ_i, and λ_0 are the Lagrange multipliers for the budget constraint, the nonnegative constraints for z_i, and the nonnegative constraint for z_0, respectively. The parameter $\mathbf{d}_{*i}$ is a vector of length n representing the ith column from the interdependency matrix $\tilde{\mathbf{D}}$.

$$\lambda = \left[\frac{\prod_{i:z_i>0} \left(\mathbf{x}^\mathsf{T}\mathbf{d}_{*i}\hat{c}_i^* k_i\right)^{1/k_i}}{\exp\left(Z - z_0 + \sum_{i:z_i>0} \frac{k_0 z_0^2}{k_i}\right)} \right]^{\left(\sum_{i:z_i>0} 1/k_i\right)^{-1}} \tag{3}$$

$$z_i = \frac{1}{k_i}\log\left(\frac{\mathbf{x}^\mathsf{T}\mathbf{d}_{*i}\hat{c}_i^* k_i}{\lambda - \lambda_i}\right) - \frac{k_0 z_0^2}{k_i} \qquad \lambda_i z_i = 0 \qquad i = 1,\ldots,m \tag{4}$$

$$-2k_0 z_0 \sum_{i=1}^{m} \mathbf{x}^\mathsf{T}\mathbf{d}_{*i}\hat{c}_i^* \exp\left(-k_i z_i - k_0 z_0^2\right) + \lambda - \lambda_0 = 0 \qquad \lambda_0 z_0 = 0 \tag{5}$$

The KKT conditions for optimality provide insight into the factors influencing the optimal allocation of resources to each industry. Equation (4) demonstrates that if some resources are allocated to industry i, z_i monotonically increases with $\mathbf{x}^\mathsf{T}\mathbf{d}_{*i}$ and $\hat{c}_i^*$. If industry i induces large impacts on the entire economic region as measured by $\mathbf{x}^\mathsf{T}\mathbf{d}_{*i}$ or if the direct impacts for that industry as measured by $\hat{c}_i^*$ are large, government officials should devote more resources to reduce losses for that industry. The optimal allocation depends on both the impacts to industry i and the interdependent impacts that industry i induces in the rest of the economy. The optimal allocation to industry i increases as k_i increases for smaller values of k_i but decreases for larger values of k_i. If allocating resources to an industry becomes more effective, the industry requires fewer resources, leaving more resources available for other industries.

We begin to solve for the KKT conditions by assuming some $z_i = 0$ and the other $z_i > 0$, which allows us to express (5) as a function of a single variable z_0. If $z_0 > 0$, (5) can be rewritten as (6) after substituting the expressions for λ in (3) and for z_i in (4).

$$\exp\left(-k_0 z_0^2\right)\left[F\exp\left(\frac{z_0 - Z}{\sum_{i:z_i>0} 1/k_i}\right)\left(1 - 2k_0 z_0 \sum_{i:z_i>0}\frac{1}{k_i}\right) - 2Gk_0 z_0\right] = 0 \tag{6}$$

where

$$F = \left[\prod_{i:z_i>0}\left(\mathbf{x}^\mathsf{T}\mathbf{d}_{*i}\hat{c}_i^* k_i\right)^{1/k_i}\right]^{\left(\sum_{i:z_i>0} 1/k_i\right)^{-1}}$$
$$G = \sum_{i:z_i=0} \mathbf{x}^\mathsf{T}\mathbf{d}_{*i}\hat{c}_i^*$$

Proposition 1 *If a given subset of the m impacted industries are individually allocated resources for recovery (i.e., $z_i > 0$ for some i) and the other industries are not individually allocated any resources (i.e., $z_i = 0$), then (6) has at most three real solutions for z_0 if $0 < z_0 < Z$.*

Proof See the Appendix.

As shown in the proof to Proposition 1, the number of solutions to (6) is determined by the number of solutions for z_0 when the expression in (7) equals 0.

$$F\exp\left(\frac{z_0 - Z}{\sum_{i:z_i>0} 1/k_i}\right)\left(1 - 2k_0 z_0 \sum_{i:z_i>0}\frac{1}{k_i}\right) - 2Gk_0 z_0 \tag{7}$$

The first derivative of (7) with respect to z_0 is given in (8).

$$F\exp\left(\frac{z_0 - Z}{\sum_{i:z_i>0} 1/k_i}\right)\left(\frac{1}{\sum_{i:z_i>0} 1/k_i} - 2k_0 z_0 - 2k_0 \sum_{i:z_i>0} \frac{1}{k_i}\right) - 2Gk_0 \tag{8}$$

As explained in Proposition 2, the budget Z must be large enough and the first derivative in (8) must first be negative and then be positive in order for (6) to have three solutions for z_0.

Proposition 2 *Equation (6) has three real solutions for z_0, where $0 \le z_0 \le Z$, if and only if the following conditions hold:*

1. $Z \ge \dfrac{F}{2Gk_0 + 2Fk_0 \sum_{i:z_i>0} 1/k_i}$
2. *The expression in (8) is less than 0 when* $z_0 = z^* = \dfrac{\frac{1}{\sum_{i:z_i>0} 1/k_i} - \sqrt{\frac{1}{\left(\sum_{i:z_i>0} 1/k_i\right)^2} - 8k_0}}{4k_0}$
3. *The expression in (8) is greater than 0 when* $z_0 = z^{**} = \dfrac{\frac{1}{\sum_{i:z_i>0} 1/k_i} + \sqrt{\frac{1}{\left(\sum_{i:z_i>0} 1/k_i\right)^2} - 8k_0}}{4k_0}$ *with* $z^{**} < Z$

Proof See the Appendix.

Each solution for z_0 leads to a unique solution for z_i and λ that satisfies (3) and (4). Given z_0, the optimization problem in (1) is convex and the KKT conditions have one unique solution. If all m industries are individually allocated resources, then $z_0 = \left(2k_0 \sum_{i=1}^{m} 1/k_i\right)^{-1}$ is the unique solution to (6) because $G = 0$. If no industries receive individual allocations, then $z_0 = Z$.

Because determining which $z_i > 0$ and $z_i = 0$ allows us to calculate z_0, we need to determine an efficient manner for selecting which $z_i > 0$ and $z_i = 0$. Although 2^m different combinations of positive and zero z_i exist, we only need to examine m possible combinations based on Proposition 3.

Proposition 3 *If it is optimal that $z_j = 0$ for some $j = 1,\ldots,m$, then $z_i = 0$ for all i such that $\mathbf{x}^\intercal \mathbf{d}_{*i} \hat{c}_i^* k_i \le \mathbf{x}^\intercal \mathbf{d}_{*j} \hat{c}_j^* k_j$.*

Proof Proof by contradiction. Assume that it is optimal that $z_j = 0$ and that $z_i > 0$ where $\mathbf{x}^\intercal \mathbf{d}_{*i} \hat{c}_i^* k_i \le \mathbf{x}^\intercal \mathbf{d}_{*j} \hat{c}_j^* k_j$. From the first-order conditions in (4), $z_j = 0$ implies that $\mathbf{x}^\intercal \mathbf{d}_{*j} \hat{c}_j^* k_j \exp\left(-k_0 z_0^2\right) < \lambda$ and $z_i > 0$ implies that $\mathbf{x}^\intercal \mathbf{d}_{*i} \hat{c}_i^* k_i \exp\left(-k_0 z_0^2\right) > \lambda$. But this cannot be true because of the original assumption. Therefore, $z_i = 0$ if $z_j = 0$. □

Based on these three propositions, we develop Algorithm 1 to find the optimal allocation for the static model. The algorithm begins by calculating the optimal allocation for z_i assuming $z_0 = 0$ (line 2). Because the optimization problem in (1) is convex in z_i, the KKT conditions in (3) and (4) are both necessary and sufficient under the assumption that $z_0 = 0$.

The algorithm continues by first assuming on line 5 that $z_i > 0$ for all $i = 1,\ldots,m$ and calculates z_0 via line 8. The set S_0 contains the industries for which $z_i = 0$, and the set S_+ contains those industries for which $z_i > 0$. As depicted in lines 10 - 13, the algorithm adds industry i_0 to the set S_0 each time through the loop where industry i_0 has the smallest product, $\mathbf{x}^\intercal \mathbf{d}_{*i} \hat{c}_i^* k_i$, of those industries currently in set S_+.

Algorithm 1: Algorithm to find optimal allocation for Model 1

Data: $\mathbf{c}^*, \mathbf{x}, \tilde{\mathbf{D}}, k_i, k_0, Z$
Result: λ, z_i, z_0
1 **begin**
2 Set $z_0 = 0$ and find unique λ and z_i that satisfy (3) and (4)
3 Store $z_0 = 0$ and z_i as a potential optimal allocation
4 $S_0 = \emptyset$
5 $S_+ = \{i : 1 \leq i \leq m\}$ and assume $z_i > 0 \; \forall i \in S_+$
6 **for** $j \leftarrow 0$ **to** $m-1$ **do**
7 **if** $j = 0$ **then**
8 $z_0 = \left(2k_0 \sum_{i=1}^{m} 1/k_i\right)^{-1}$
9 **else**
10 $i_0 = \underset{i \in S_+}{\operatorname{argmin}} \{\mathbf{x}^\intercal \mathbf{d}_{*i} \hat{c}_i^* k_i\}$
11 $S_0 = S_0 \cup \{i_0\}$
12 $S_+ = S_+ \setminus \{i_0\}$
13 Assume $z_i = 0 \; \forall i \in S_0$ and $z_i > 0 \; \forall i \in S_+$
14 **if** *the conditions of Proposition 2 are satisfied* **then**
15 Use (6) to find three solutions for z_0
16 **else if** $F \exp\left(1 - 2k_0 Z \sum_{i:z_i>0} \frac{1}{k_i}\right) - 2Gk_0 Z > 0$ **then**
17 Use (6) to find two solutions for z_0 assuming a solution exists
18 If no solution exists for z_0 return to line 6
19 **else**
20 Use (6) to find one solution for z_0
21 For each z_0 that satisfies (6), use (3) to solve for λ and (4) to solve for z_i
22 **if** $\exists i \in S_0$ *such that* $z_i > 0$ *or* $\exists i \in S_+$ *such that* $z_i \leq 0$ **then**
23 Proposed solution does not satisfy KKT conditions
24 **else**
25 Store λ, z_i, and z_0 as a potential optimal allocation
26 Store $z_0 = Z$ and $z_i = 0$, $i = 1, \ldots, m$ as a potential optimal allocation
27 Compare all potential optimal allocations from lines 3, 25, and 26 and choose allocation that minimizes objective function in (1)

Lines 14 - 20 calculate z_0 so that (6) is satisfied, and it checks to see whether one, two, or three solutions exist. Proposition 2 is used to check whether three solutions exist. If the conditions of Proposition 2 are not met, one solution exists if (6) is negative when $z_0 = Z$ and either two or no solution exist if (6) is positive when $z_0 = Z$. For each calculated value of z_0, the algorithm uses (3) to calculate λ given z_0 and uses (4) to calculate z_i given z_0 and λ in line 21. As lines 22 and 23 depict, if (4) calculates $z_i > 0$ for any i in S_0 or calculates $z_i \leq 0$ for any i in S_+, then those values for z_0 and z_i cannot be an optimal allocation, and they are discarded. The algorithm also stores $z_0 = Z$ as a potential solution. Finally, line 27 compares all the allocations that satisfy the KKT conditions and selects the allocation that minimizes the objective function in (1).

3.3 Model 2: Discrete-time dynamic allocation

Disruptions can last a period of time, and recovering from a disruption often requires allocating resources over time. A discrete-time dynamic resource allocation model is given in (9) where a policy maker allocates resources at fixed points in time. The disruption occurs at time $t = 0$, and $t = t_f$ is the fixed final time in the model. The model assumes it takes one time period for the allocated resources to reduce the industry impacts. Therefore, the policy maker seeks to minimize the total production losses (both direct and indirect) in the time interval $[1, t_f]$ by allocating resources in the time interval $[0, t_f - 1]$. The other variables in this model correspond to those in the static allocation model except that many of the variables change over time.

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{t=1}^{t_f} \mathbf{x}(t)^{\mathsf{T}} \tilde{\mathbf{D}} \mathbf{c}^*(t) \\
\text{subject to} & c_i^*(t+1) = c_i^*(t) \exp\left(-k_i(t) z_i(t) - k_0(t) z_0^2(t)\right) \\
& \qquad\qquad\qquad i = 1, \ldots, m \quad t = 0, \ldots, t_f - 1 \\
& \displaystyle\sum_{t=0}^{t_f-1} \left[z_0(t) + \sum_{i=1}^{m} z_i(t) \right] \leq Z \\
& z_0(t), z_i(t) \geq 0 \quad i = 1, \ldots, m \quad t = 0, \ldots, t_f - 1 \\
& c_i^*(0) = \hat{c}_i^* \qquad i = 1, \ldots, m
\end{array}
\tag{9}
$$

Because resources allocated over the entire time interval are constrained by the overall budget Z, the optimal decision may be to allocate the entire budget in the first time period, $t = 0$, or spread the resources over time. This timing decision depends on how the effectiveness of allocation changes over time as governed by $k_i(t)$ and $k_0(t)$. If each $k_i(t)$ and $k_0(t)$ remain constant over time or decrease with time, a policy maker should allocate the entire budget Z at time $t = 0$. The optimal allocation follows that of the static allocation model.

If $k_i(t)$ or $k_0(t)$ increases with time, it may be optimal to wait to allocate some of the available resources. A trade-off exists between allocating resources so that recovery begins immediately and saving resources in order to impact recovery the most. Although policy makers may also hold resources in reserve because they are uncertain about how the disruption will develop, the proposed dynamic model is deterministic. It assumes the decision maker knows perfectly what will happen in the future. A deterministic model provides important insight about allocating resources and is a useful step before building an accurate stochastic model. Future extensions of the model will examine the impact of uncertainty on optimal resource allocation after a disruption.

Solving the dynamic model using standard recursive relationships becomes impractical because of the curse of dimensionality. At the last decision period $t = t_f - 1$, the direct impacts for each of the m industries can range between 0 and $\hat{c}_i^*$ and the remaining budget can range between 0 and Z. Determining the optimal allocation for each of the possible direct impacts and budget at $t_f - 1$ to inform the decision in the previous period $t = t_f - 2$ requires a large number of calculations. Starting at time $t = 0$ and moving forward fails because the allocation considering only the current time period is not always optimal when considering multiple time periods.

A branch-and-bound algorithm (see Lawler and Wood 1966; Horst and Tuy 1990) can rely on a convex relaxation of the dynamic problem in (9) to find an allocation of resources such that production losses are within a desired $\varepsilon > 0$ of the minimum value of (9). The minimum value from solving the optimization problem in (10) serves as a lower bound on the minimum value in (9) where $\underline{b}_t \geq 0$ is a lower bound on $z_0(t)$ and $\bar{b}_t \leq Z$ is an upper bound on $z_0(t)$.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{t=1}^{t_f} \mathbf{x}(t)^\mathsf{T} \tilde{\mathbf{D}} \mathbf{c}^*(t) \\
\text{subject to} \quad & c_i^*(t+1) = c_i^*(t) \exp\left(-k_i(t) z_i(t) - k_0(t) \bar{b}_t z_0(t)\right) \\
& \qquad\qquad\qquad i = 1, \ldots, m \quad t = 0, \ldots, t_f - 1 \\
& \sum_{t=0}^{t_f-1} \left[z_0(t) + \sum_{i=1}^{m} z_i(t) \right] \leq Z \\
& \underline{b}_t \leq z_0(t) \leq \bar{b}_t \qquad t = 0, \ldots, t_f - 1 \\
& z_i(t) \geq 0 \qquad i = 1, \ldots, m \quad t = 0, \ldots, t_f - 1 \\
& c_i^*(0) = \hat{c}_i^* \qquad i = 1, \ldots, m
\end{aligned}
\tag{10}
$$

Proposition 4 *The optimal value in (10) provides a lower bound on the optimal value in (9) given that each $z_0(t)$ is bounded above by $\bar{b}_t$ and below by $\underline{b}_t$.*

Proof Let $\tilde{z}_0(t)$ and $\tilde{z}_i(t)$ be a feasible solution to both (9) and (10) where $\underline{b}_t \leq \tilde{z}_0(t) \leq \bar{b}_t$. Notice that if the constraint $\underline{b}_t \leq z_0(t) \leq \bar{b}_t$ is added to the optimization problem in (9), a feasible solution to one problem is also feasible in the other problem.

Clearly, $c_i^*(t) \exp\left(-k_i(t) \tilde{z}_i(t) - k_0(t) \bar{b}_t \tilde{z}_0(t)\right) \leq c_i^*(t) \exp\left(-k_i(t) \tilde{z}_i(t) - k_0(t) \tilde{z}_0^2(t)\right)$ for $i = 1, \ldots, m$ and $t = 0, \ldots, t_f - 1$. Because the direct impacts in (10) as given by $c^*(t+1)$ are less than or equal to the direct impacts in (9) for each industry and in each time period, the objective function in (10) is less than or equal to the objective function in (9) for any feasible allocation decision. Thus, the optimal value in (10) provides a lower bound for (9). □

The optimization problem in (10) is convex. Computer optimization programs can solve the optimization problem using any one of a number of popular algorithms such as interior-point methods (Byrd et al. 1999; Waltz 2006) or sequential quadratic programming (Nocedal and Wright 1999). Algorithm 2 presents a branch-and-bound algorithm that solves (10) for different bounds $\left[\underline{b}_t, \bar{b}_t\right]$ on $z_0(t)$. The algorithm begins by solving (10) with the largest bounds possible, $\underline{b}_t = 0$ and $\bar{b}_t = Z$ for each $z_0(t)$, $t = 1 \ldots t_f - 1$ (lines 2 and 3). Calculating the value of the objective function from (9) with the optimal allocation from (10) establishes an upper bound UB^* on the minimum production lost in the dynamic model (line 4). These bounds are stored in the set W_0 where $\underline{\mathbf{b}}$ and $\bar{\mathbf{b}}$ are each vectors of length t_f representing the lower bounds and the upper bounds (in this case 0 and Z), respectively, on each $z_0(t)$.

Each set of bounds stored in W_0 are divided into H new bounds where $H \geq 1$. (If $H = 1$, the algorithm keeps the original bounds.) Initially, W_0 only contains one set of bounds $\underline{\mathbf{b}} = \mathbf{0}$ and $\bar{\mathbf{b}} = \mathbf{Z}$ where $\mathbf{0}$ and $\mathbf{Z}$ are each vectors of length t_f. As depicted in line 11, the new bounds $\left[\underline{b}_t^{(h)}, \bar{b}_t^{(h)}\right]$, $h = 1, \ldots, H$ must be chosen such that the union of these new bounds spans the space covered by W_0. For example, if $W_0 = [\mathbf{0}, \mathbf{Z}]$, $H = 3$, and $t_f = 4$, the three sets might be: (i) $\left[\underline{\mathbf{b}}^{(1)}, \bar{\mathbf{b}}^{(1)}\right] = \left[(0,0,0,0), \left(Z/3, Z, Z, Z\right)\right]$; (ii) $\left[\underline{\mathbf{b}}^{(2)}, \bar{\mathbf{b}}^{(2)}\right] = \left[\left(Z/3, 0,0,0\right), \left(2Z/3, Z, Z, Z\right)\right]$; and (iii) $\left[\underline{\mathbf{b}}^{(3)}, \bar{\mathbf{b}}^{(3)}\right] = \left[\left(2Z/3, 0,0,0\right), (Z, Z, Z, Z)\right]$. The union of these three sets is $[(0,0,0,0), (Z,Z,Z,Z)]$, which equals W_0. The only bounds that are branched or divided are those corresponding to time $t = 0$. We may want to further divide the first set of bounds $\left[(0,0,0,0), \left(Z/3, Z, Z, Z\right)\right]$ during a subsequent time through the loop. If we select $H = 2$, we may divide the bounds corresponding to time $t = 1$: (i) $\left[(0,0,0,0), \left(Z/3, Z/2, Z, Z\right)\right]$ and (ii) $\left[\left(0, Z/2, 0, 0\right), \left(Z/3, Z, Z, Z\right)\right]$. A different but equally viable division corresponds to $t = 0$: (i) $\left[(0,0,0,0), \left(Z/6, Z, Z, Z\right)\right]$ and (ii) $\left[\left(Z/6, 0,0,0\right), \left(Z/3, Z, Z, Z\right)\right]$.

Algorithm 2: Branch-and-bound algorithm to calculate optimal allocation for Model 2

Data: $\mathbf{c}^*(0), \mathbf{x}(t), \tilde{\mathbf{D}}, t_f, k_i(t), k_0(t), Z, \varepsilon$
Result: $z_i(t), z_0(t), Q$
1 **begin**
2 Set $\underline{b}_t = 0$ and $\bar{b}_t = Z$ for $t = 0, \ldots, t_f - 1$
3 Solve (10) with bounds $[\underline{B}_t, \bar{B}_t]$ for each $z_0(t)$
4 Input the optimal solution from (10) into the objective function in (9) and set UB^* equal to the objective function's value
5 $W_0 = [\underline{\mathbf{b}}, \bar{\mathbf{b}}]$
6 $J = 1, l = 0$
7 **repeat**
8 $l = 0, W = \emptyset, UB_W = \emptyset, LB_W = \emptyset$
9 **for** $j \leftarrow 1$ **to** J **do**
10 Choose $H \geq 1$ as the number of branches
11 Establish bounds for each branch such that $\underline{b}_t^{(h)} \leq z_0(t) \leq \bar{b}_t^{(h)}$ where $h = 1, \ldots, H$. The bounds for each branch must be chosen such that $W_0(j) \subseteq \bigcup_{h=1}^{H} \left\{ \left[\underline{\mathbf{b}}^{(h)}, \bar{\mathbf{b}}^{(h)} \right] \right\}$
12 **for** $h \leftarrow 1$ **to** H **do**
13 Solve (10) with bounds $\left[\underline{b}_t^{(h)}, \bar{b}_t^{(h)} \right]$ for each $z_0(t)$
14 Set $LB(h)$ equal to the optimal value of (10)
15 **if** $LB(h) < UB^*$ **then**
16 $l = l + 1$
17 Input the optimal solution from (10) into the objective function in (9) and assign $UB(h)$ equal to the objective function's value
18 $W(l) = \left[\underline{\mathbf{b}}^{(h)}, \bar{\mathbf{b}}^{(h)} \right]$
19 $UB_W(l) = UB(h)$
20 $LB_W(l) = LB(h)$
21 $l^* = \text{argmin}\{UB_W\}$
22 $UB^* = UB_W(l^*)$
23 **for** $h \leftarrow 1$ **to** l **do**
24 **if** $LB_W(h) \geq UB^*$ **then**
25 Remove $\left[\underline{\mathbf{b}}^{(h)}, \bar{\mathbf{b}}^{(h)} \right]$ from W
26 Set J equal to the number of bounds in W
27 $W_0 = W$
28 **until** $UB^* - \min\{LB_W\} < \varepsilon$;
29 Set $z_i(t)$ and $z_0(t)$ equal to the optimal solution corresponding to l^*
30 $Q = UB^*$

The algorithm loops through each of the H branches in lines 12 - 20. A lower bound on production losses $LB(h)$ is calculated each time through the loop using the bounds $\left[\underline{\mathbf{b}}^{(h)}, \bar{\mathbf{b}}^{(h)} \right]$ in (10). If $LB(h) > UB^*$, then $\left[\underline{\mathbf{b}}^{(h)}, \bar{\mathbf{b}}^{(h)} \right]$ does not contain the optimal allocation for $z_0(t)$, and the algorithm prunes or eliminates this branch from consideration. Otherwise, an upper bound on production losses $UB(h)$ is also calculated, and the bounds are added to the set W. The upper bound and lower bound on production losses that correspond to each bound on $z_0(t)$ are stored in UB_W and LB_W, respectively.

After all the bounds in W_0 have been subdivided into new bounds, which are stored in W, UB^* is set to equal the minimum upper bound in UB_W (lines 21 and 22). The algorithm removes any bounds in W whose lower bound LB is greater than or equal to UB^*. After these bounds are removed from W, line 27 sets W_0 equal to W, and the algorithm repeats the process beginning on line 9 by creating branches for each of the bounds in W_0. The loop ends when the difference between UB^* and the minimum lower bound is less than ε, which indicates the gap between the minimum values in (9) and (10) is less than ε. As the bounds get narrower, the algorithm converges to the optimal allocation. If $\underline{b}_t = \bar{b}_t = z_0^*(t)$, where $z_0^*(t)$ is the optimal allocation for (9), the objective function from (10) equals the minimum value in the original dynamic model. The algorithm returns $Q = UB^*$ as the minimum value for (9) and $z_i(t)$ and $z_0(t)$ corresponding to UB^* as the optimal allocation.

Exploring and subdividing all the possible branches until the gap between the minimum upper bound and minimum lower bound is less than ε can take a long time. The algorithm takes 20 to 30 minutes to solve the *Deepwater Horizon* application in the next section. The algorithm can be modified so that it always explores a branch with the smallest lower bound in the hope that subdividing this branch will generate a smaller lower upper bound.

4 Application: *Deepwater Horizon* oil spill

The resource allocation models are applied to a case study examining the economic impacts of the *Deepwater Horizon* oil spill. As a result of the April 20, 2010 explosion on the *Deepwater Horizon* oil rig, almost 5 million barrels of crude oil spilled into the Gulf of Mexico until the leak was finally capped on July 15. BP, which operated the oil rig, agreed to establish a $20 billion fund to pay for the damage to the Gulf ecosystem, reimburse state and local governments for the cost of responding to the spill, and compensate individuals for lost business. This application measures the lost production in the region due to the spill's direct impacts on five different industries. Parameter estimation for the resource allocation models derive from publicly available economic data, think-tank and government reports, journal articles, and news stories.

4.1 Assumptions and parameter estimation

The models include five Gulf states (Texas, Louisiana, Mississippi, Alabama, and Florida). The U.S. Bureau of Economic Analysis (2010a,b, 2011) collects economic data used to populate production for each industry in those states (the vector $\mathbf{x}$) and the interdependencies among industries (the matrix $\tilde{\mathbf{D}}$ in Table 1). The models combine the five Gulf States into a single economy with a total of $n = 63$ industries.

The models focus exclusively on production losses due to inoperable facilities or reduced demand and ignore the severe environmental damage. Direct impacts from the oil spill include: (i) demand losses because consumers decide to buy or consume fewer goods and services as a result of the oil spill and (ii) less industry production because facilities are inoperable. Demand losses occurred because people did not travel to the Gulf for vacation or buy fish from the Gulf (and fewer fish were caught). Firms drilled for less oil in the Gulf because of the moratorium, the lack of new leases and licenses, and the need for enhanced safety measures. The models consider that the oil spill directly impacted the Fishing and Forestry, Real Estate, Amusements, Accommodations, and Oil and Gas industries ($m = 5$).

The decision maker for this application is a hypothetical policy maker responsible for limiting economic losses in the five Gulf states. The policy maker controls resources that can be used to increase demand for seafood, tourism, and real estate in the Gulf, implement new safety requirements in the offshore oil platforms, and remove crude oil from the Gulf which benefits all of the impacted industries. Although the U.S. federal government has

Table 1 Interdependence matrix $\tilde{\mathbf{D}}$ for the Gulf region

Industry	Fishing and Forestry	Real Estate	Amuse-ments	Accommo-dations	Oil and Gas
Fishing and Forestry	1.120	0.026	0.005	0.013	0.005
Real Estate	< 0.001	1.057	0.003	0.004	0.002
Amusements	< 0.001	0.004	1.000	0.002	0.001
Accommodations	< 0.001	0.022	0.002	1.005	0.003
Oil and Gas	0.001	0.011	0.002	0.004	1.067
Farms	0.003	0.002	0.002	0.002	0.001
Mining	0.001	0.033	0.006	0.005	0.023
Mining Support	< 0.001	0.001	< 0.001	< 0.001	0.042
Utilities	< 0.001	0.025	0.004	0.012	0.010
Construction	< 0.001	0.024	0.001	0.002	0.021
Wood Products	0.004	0.092	0.003	0.011	0.016
Nonmetallic Mineral Products	0.001	0.056	0.002	0.004	0.022
Primary Metals	0.001	0.022	0.003	0.004	0.043
Fabricated Metal Products	0.002	0.027	0.002	0.005	0.034
Machinery	0.002	0.014	0.001	0.002	0.021
Computer and Electronic Products	< 0.001	0.011	0.001	0.005	0.004
Electrical Equipment	0.002	0.023	0.002	0.005	0.012
Motor Vehicles	0.001	0.005	0.001	0.001	0.005
Other Transportation Equipment	< 0.001	0.002	< 0.001	0.001	0.001
Furniture Products	< 0.001	0.076	0.004	0.003	0.006
Misc. Manufacturing	< 0.001	0.006	0.001	0.003	0.004
Food, Beverage, and Tobacco	0.001	0.002	0.003	0.004	0.001
Textile Mills	0.002	0.013	0.006	0.005	0.005
Apparel and Leather	0.001	0.009	0.002	0.003	0.003
Paper Products	0.001	0.015	0.003	0.018	0.009
Printing	0.001	0.023	0.006	0.016	0.006
Petroleum and Coal Products	0.001	0.007	0.001	0.002	0.004
Chemical Products	0.004	0.011	0.003	0.002	0.016
Plastics and Rubber Products	0.001	0.023	0.002	0.004	0.015
Wholesale Trade	0.001	0.009	0.001	0.003	0.007
Retail Trade	< 0.001	0.010	0.001	0.001	0.002
Air Transportation	< 0.001	0.016	0.002	0.004	0.003
Rail Transportation	0.001	0.019	0.003	0.005	0.022
Water Transportation	< 0.001	0.001	< 0.001	0.001	0.002
Truck Transportation	0.001	0.015	0.004	0.004	0.008
Ground Passenger Transportation	< 0.001	0.022	0.002	0.005	0.003
Pipeline Transportation	< 0.001	0.011	0.002	0.005	0.065
Other Transportation	0.001	0.014	0.004	0.005	0.006
Warehousing and Storage	0.001	0.015	0.004	0.009	0.006
Publishing	< 0.001	0.012	0.002	0.006	0.005
Motion Picture and Sound Recording	< 0.001	0.009	0.003	0.017	0.002
Broadcasting and Telecommunications	< 0.001	0.021	0.004	0.009	0.006
Information and Data Processing	0.001	0.274	0.004	0.006	0.011
Federal Reserve Banks	< 0.001	0.064	0.002	0.003	0.005
Securities, Commodity Contracts, and Investments	< 0.001	0.091	0.007	0.009	0.004
Insurance Carriers	< 0.001	0.004	< 0.001	< 0.001	< 0.001
Funds, Trusts, and Other Financial Vehicles	0.001	0.013	0.003	0.008	0.076
Rental and Leasing Services	0.001	0.065	0.006	0.016	0.011
Legal Services	< 0.001	0.018	0.002	0.005	0.009
Computer Systems Design	0.001	0.038	0.007	0.018	0.013
Misc. Professional Services	0.001	0.028	0.007	0.031	0.031
Management of Companies	< 0.001	0.066	0.006	0.016	0.007
Administrative and Support Services	< 0.001	0.178	0.007	0.018	0.008
Waste Management Services	0.001	0.003	0.001	0.001	0.001
Educational Services	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001
Ambulatory Health Care Services	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001
Hospitals and Nursing	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001
Social Assistance	< 0.001	0.024	0.008	0.013	0.007
Performing Arts, Sports, and Museums	< 0.001	0.011	0.001	0.006	0.001
Food Services	< 0.001	0.030	0.003	0.006	0.003
Other Services	< 0.001	0.003	< 0.001	0.003	0.001
Federal Government	< 0.001	0.006	0.001	0.002	0.001
Sum	1.166	2.754	1.170	1.386	1.733

Table 2 Input values for *Deepwater Horizon* application

i	Industry	k_i (per \$1 million)	$\hat{c}_i^*$
1	Fishing and Forestry	0.074	0.0084
2	Real Estate	0	0.047
3	Amusements	0.0038	0.21
4	Accommodations	0.0027	0.16
5	Oil and Gas	0.0057	0.079
	All industries simultaneously	$k_0 = 7.4 * 10^{-9}$ (per \$1 million^2)	

responsibility for many of these areas, in practice, the federal government, state and local entities, and the private sector all control resources that can be used for these types of tasks.

Table 2 displays the parameter estimates for the effectiveness of allocating resources, k_i, and the direct impacts for each industry, $\hat{c}_i^*$. Allocating resources to one of the industries directly impacted by reduced demand means better communication about the risks, safety, and cleanliness of the products and services produced by these industries. The models assume that these resources can be expressed in monetary terms. If people are not consuming fish caught in the Gulf of Mexico, resources can be devoted to testing fish for oil contamination and to a public relations campaign explaining that fish are safe for consumption. A decrease of $63 million in fishing revenue due to the oil spill (National Resources Defense Council 2011) enables the direct impacts for the Fishing and Forestry industry ($i = 1$) to be estimated. The parameter k_1 is derived from two studies (Richards and Patterson 1999; Verbeke and Ward 2001) examining the effect of positive media stories following two different food scares.

The direct impacts for Amusements ($i = 3$) and Accommodations ($i = 4$) are based on an estimate that tourism declined in Louisiana, Alabama, Mississippi, and Florida by 30% although tourism in Texas does not appear to have been impacted (Market Dynamics Research Group 2010; Oxford Economics 2010). Tourism to the Gulf can be encouraged by ensuring that the beaches are free of oil and debris and demonstrating to potential tourists that the beaches are safe and open. The effectiveness parameters are derived from an Oxford Economics (2010) study that argues for a return on investment of 15 to 1 in tourism marketing. For the Real Estate industry ($i = 2$), the models assume that the demand for housing in the four states fell 10% and that increasing demand for housing depends entirely on tasks devoted to helping all industries such as stopping the oil leak and cleaning up the oil. Hence, $k_2 = 0$.

Allocating resources to the Oil and Gas industry ($i = 5$) means implementing new safety measures to reduce the risk of an accident on an offshore oil platform. The U.S. government imposed a six-month moratorium on deepwater drilling in the Gulf of Mexico, and it did not issue new leases for oil exploration in the Gulf until December 2011 (Fowler 2011). Spending more to improve the safety of deepwater drilling may have induced the federal government to lift the moratorium earlier and grant more licenses and leases. Direct impacts are based on domestic oil production from the Gulf of Mexico in 2010 (U.S. Energy Information Administration 2011), and k_5 is derived from an estimate that the new safety measures cost $183 million (McAndrews 2011).

Capping the oil leak, containing the spill, and removing crude oil from the ocean can simultaneously benefit all five directly impacted industries. If less oil spills or if the oil is cleaned up more quickly, people are more likely to eat fish from the Gulf and vacation on

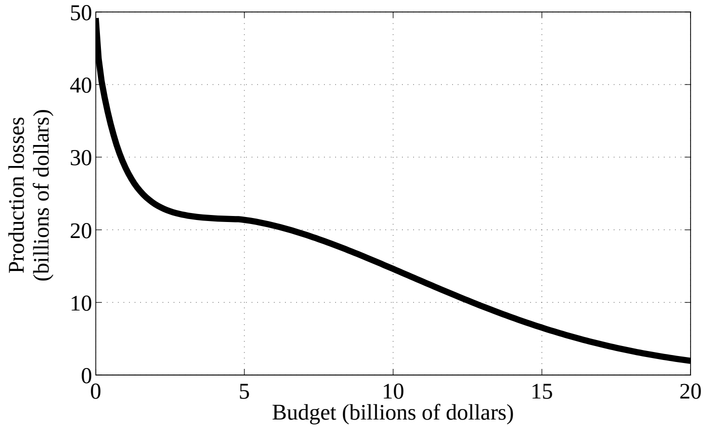

Fig. 1 Production losses for Model 1 at different budget amounts

its beaches. The Oil and Gas industry can also benefit because lifting the moratorium is less politically sensitive if the consequences of the oil spill are limited. Approximately \$11.6 billion was spent on stopping the oil leak and cleaning up the oil (Trefis Team 2011), and k_0 (per \$1 million squared) is estimated by assuming that $\sqrt{k_0} * \$11600 = 1$. This assumption implies that billions of dollars must be allocated in order to substantially reduce the direct impacts on the five industries.

4.2 Model 1 results

Figure 1 depicts total production losses generated by using Algorithm 1 to solve the static allocation model for different budgets ranging from \$0 to \$20 billion, where \$20 billion reflect the amount in BP's fund for reimbursing cleanup costs and lost business. Production losses total \$49.1 billion if no resources are allocated and drop to \$2.0 billion if the budget is \$20 billion.

Table 3 shows the optimal allocation for each industry for five different budget amounts: \$1, \$4.8, \$5, \$10, and \$20 billion. As implied by the table, if the budget is less than or equal to \$4.8 billion, the policy maker should not devote any resources to simultaneously help all industries because these industries do not benefit as much as they do from each one being targeted individually. If the budget is \$4.8 billion, Accommodations should receive the largest share because the money is less effective for this industry but the direct impacts are very large. The policy maker should spend \$1.5 and \$1.2 billion to help Amusements and Oil and Gas but only \$59 million for Fishing and Forestry because the direct impacts in this latter industry are less severe and the resources are very effective.

Because spending money to help all industries recover simultaneously only becomes optimal when the budget is \$4.9 billion or greater, the shape of the curve in Figure 1 changes at \$4.9 billion. If no money is ever spent to help all industries recover, production losses will not drop below \$20 billion even if the budget is \$20 billion, and the figure demonstrates that the curve begins to flatten out when the budget is \$4 billion. When the budget increases to \$4.9 billion, the shape of the curve changes and production losses drop significantly because resources are allocated to help all industries. Production losses continue to decrease at a

Table 3 Optimal allocation amounts for Model 1

Industry	Millions of dollars allocated to each industry				
Fishing and Forestry	0	59	34	12	0
Real Estate	0	0	0	0	0
Amusements	250	1,458	968	543	278
Accommodations	379	2,107	1,407	799	420
Oil and Gas	372	1,176	850	567	391
All industries simultaneously	0	0	1,741	8,079	18,911
Total budget	1,000	4,800	5,000	10,000	20,000

steep rate because the model squares z_0. Consequently, proportionally more resources should be allocated to help all industries as the budget increases. As shown in Table 3, almost 95% of a $20 billion budget should be spent on this category. As discussed in Section 3, squaring z_0 reflects the assumption that as more money is allocated to stop the oil spill and clean the oil, these tasks become easier, the work is accomplished more quickly, and the affected industries suffer less. This unique modeling assumption appears reasonable for a budget less than $20 billion with the given parameters. If the budget were larger than $20 billion, almost all of the money would be allocated to all industries because it becomes so effective. We will revisit this assumption by performing sensitivity analysis on the power to which z_0 is raised is varied.

4.3 Model 2 results

The effects from major disruptions can last several months or even years, and the Coast Guard and BP engineers worked for almost three months to stop the oil leak. Government officials working to contain and recover from disruptions need to make decisions at different points in time. The discrete-time dynamic model discussed previously can provide guidance on the optimal way to allocate resources over time. This model analyzes the oil spill for one year and divides the year into 12 months, and $t_f = 12$. Regional production is assumed to be constant in each month, and $\mathbf{x}(t) = \mathbf{x}/12$.

If the effectiveness of allocating resources, $k_i(t)$ and $k_0(t)$, decreases or remains constant with time, the policy maker should allocate all resources at time $t = 0$ according to the optimal division suggested by the results from Model 1. Encouraging people to eat fish caught in the Gulf and to vacation on the beaches may become more effective with time because people will worry less about the risks. The effectiveness of allocating resources to individual industries is assumed to increase linearly with time, and $k_i(t) = (t+1)k_i$ for $t = 0, \ldots, t_f - 1$ and $i = 1, \ldots, 5$. Tasks such as stopping the oil leak and removing crude oil from the Gulf may not become more effective as time passes, and $k_0(t) = k_0$ for all t. Although these tasks may get easier as more oil is removed, the effectiveness remains independent of time. Because $k_0(t)$ remains constant over time, it is never optimal to spend money on $z_0(t) = 0$ for time $t > 0$.

Algorithm 2 solves Model 2 for four different budgets. The algorithm solves the dynamic problem so that the upper and lower bounds on production losses are within $\varepsilon = \$1$ million of each other. Production losses exceed $1 billion dollars, so the value of ε chosen guarantees that the solution provided by the algorithm is within 0.1 percent of the minimum production losses. We use Matlab version R2012a fmincon program and its sequential quadratic programming algorithm to solve the convex problem in (10) in order to calculate the lower bound for each branch. Figure 2 depicts the first three levels on the bounds $\left[\underline{b}_t, \bar{b}_t\right]$ for $z_0(t)$ when the total budget is \$5 billion. We initially choose $H = 10$ branches, and the second iteration in the figure displays $H = 3$ subbranches for each unpruned branch.

Results from Model 2 reveal that most of the resources should be allocated to benefit all industries simultaneously at time $t = 0$ if the budget is \$5 billion or more (Table 4). If the budget is \$1 billion, no resources should be allocated to help all industries because the amount of money that could be spent is too small to make a difference. Similar to Model 1, this difference in the amount of resources that should be allocated to all industries reflects our modeling assumption of squaring z_0. Proportionally more money should be spent to benefit all industries as the budget increases. All the money in this category should be spent in the first time period because the effectiveness of allocating to all industries remains constant over time. Spending billions of dollars in one month may be impossible in reality, and an extension of this model could include an additional constraint that establishes a maximum amount that could be spent in a single time period. Even with such a constraint, stopping the spill and cleaning up the oil should be accomplished as quickly as possible, as Model 2 demonstrates. The remainder of the budget should be spent on the other industries during the first five time periods, with most of the money being allocated during the first two time periods. The deterministic nature of Model 2 ignores any incentive the decision maker may have to hold money in reserve to learn more about how the oil spill will impact industries.

Given the assumptions that $z_0(t)$ is squared and that $k_0(t)$ remains constant over time, Model 2's results emphasize that stopping and containing the oil spill should be the top priority, especially for large budgets. When a disruption occurs, allocating substantial resources to help individual industries recover is suboptimal if the disruption (e.g., an oil spill) is worsening. Some resources should remain in reserve to help specific industries recover once the primary disruption or spill is contained. The majority of the resources for individual industries should be allocated in the first few time periods, and all entire budget should be allocated before time $t = 6$. If the model of a disruption relies on different assumptions, it may be optimal to allocate more resources to individual industries and/or spend more money in future time periods.

4.4 Sensitivity analysis

Sensitivity analysis on a few key parameters and on an important model assumption provides insight into how these parameters and assumptions affect the optimal allocation of resources. Sensitivity analysis is explored on the the direct impacts and the effectiveness of allocating resources to the Fishing and Forestry industry, the effectiveness of allocating resources to all industries, and the modeling assumption that z_0 is squared. We perform the sensitivity analysis on Model 1 although the conclusions can also be applied to Model 2.

The base case results for both models recommend allocating less than \$60 million to the Fishing and Forestry industry ($i = 1$). Sensitivity analysis can reveal if this recommendation remains valid if the allocation effectiveness, k_1, and direct impacts, $\hat{c}_1^*$, change (Figure 3). The optimal allocation to this industry increases as $\hat{c}_1^*$ increases. As k_1 increases, the optimal allocation initially increases but then decreases. Although the value of k_1 at which the policy maker should allocate the most to Fishing and Forestry depends on c_1^*, the optimal allocation decreases for $k_1 > 0.005$ for any value of $\hat{c}_1^*$. For a \$10 billion budget, the most the industry should receive is \$710 million at $k_1 = 0.015$ and $\hat{c}_1^* = 0.5$. This extreme level of direct

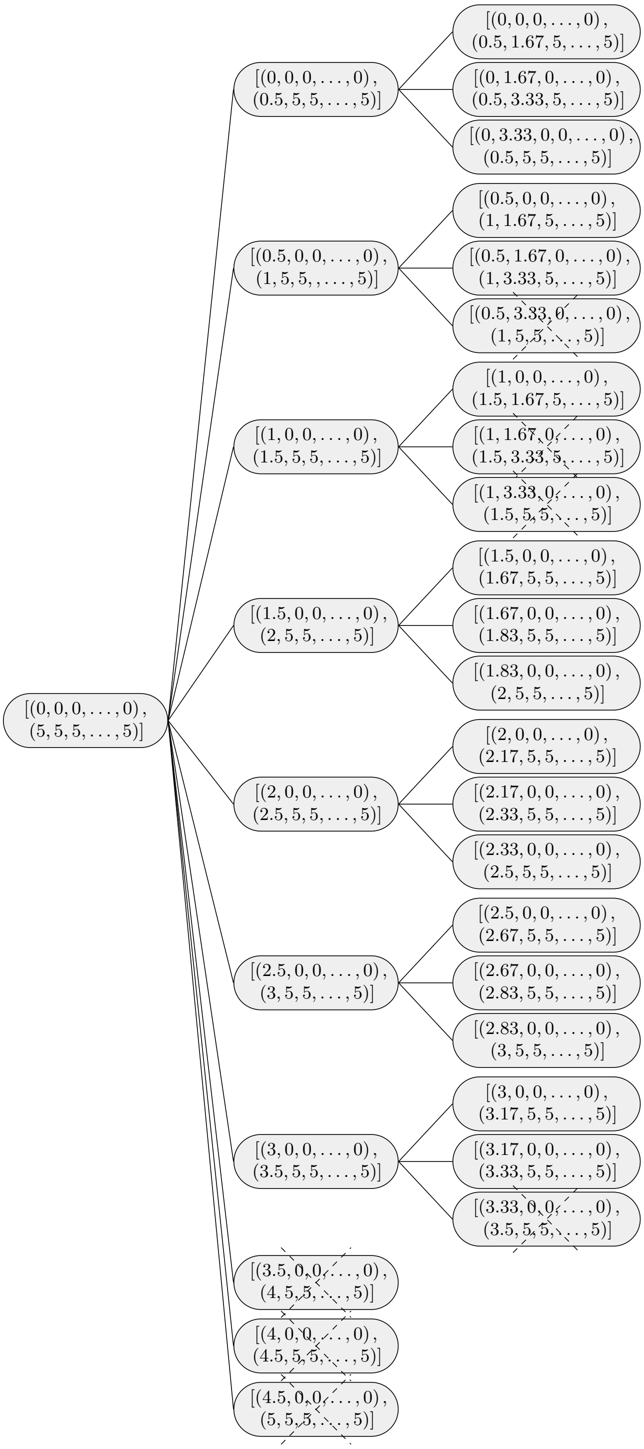

Fig. 2 An example of the first three levels on the bounds for $z_0(t)$ for a budget of \$5 billion. All numbers are in billions of dollars. A dashed X means the branch has been pruned because the branch's lower bound exceeds the best upper bound.

Table 4 Optimal allocation amounts for Model 2 (millions of dollars allocated to each industry)

$1 billion budget, $24.8 billion in lost production

Months since disruption	0	1	2	3	4	5	6	7	8	9	10	11
Fishing and Forestry	0	0	2	2	1	1	0	0	0	0	0	0
Real Estate	0	0	0	0	0	0	0	0	0	0	0	0
Amusements	0	125	61	34	21	15	0	0	0	0	0	0
Accommodations	0	190	87	48	30	21	0	0	0	0	0	0
Oil and Gas	180	96	40	22	14	10	0	0	0	0	0	0
All industries simultaneously	0	0	0	0	0	0	0	0	0	0	0	0

$5 billion budget,$20.1 billion in lost production

Months since disruption	0	1	2	3	4	5	6	7	8	9	10	11
Fishing and Forestry	3	7	3	2	1	1	0	0	0	0	0	0
Real Estate	0	0	0	0	0	0	0	0	0	0	0	0
Amusements	375	144	61	34	21	15	0	0	0	0	0	0
Accommodations	558	206	87	48	31	21	0	0	0	0	0	0
Oil and Gas	455	96	40	22	14	10	0	0	0	0	0	0
All industries simultaneously	2,745	0	0	0	0	0	0	0	0	0	0	0

$10 billion budget, $13.5 billion in lost production

Months since disruption	0	1	2	3	4	5	6	7	8	9	10	11
Fishing and Forestry	0	1	3	2	1	1	0	0	0	0	0	0
Real Estate	0	0	0	0	0	0	0	0	0	0	0	0
Amusements	49	144	61	34	21	15	0	0	0	0	0	0
Accommodations	92	206	87	48	31	21	0	0	0	0	0	0
Oil and Gas	238	96	40	22	14	10	0	0	0	0	0	0
All industries simultaneously	8,763	0	0	0	0	0	0	0	0	0	0	0

$20 billion budget, $1.7 billion in lost production

Months since disruption	0	1	2	3	4	5	6	7	8	9	10	11
Fishing and Forestry	0	0	0	1	1	1	0	0	0	0	0	0
Real Estate	0	0	0	0	0	0	0	0	0	0	0	0
Amusements	0	52	61	34	21	15	0	0	0	0	0	0
Accommodations	0	86	87	48	30	21	0	0	0	0	0	0
Oil and Gas	83	96	40	22	14	10	0	0	0	0	0	0
All industries simultaneously	19,276	0	0	0	0	0	0	0	0	0	0	0

impacts is very unlikely, however, and $710 million still only represents 7.1% of the entire budget. As the effectiveness increases, even less money needs to be allocated to the Fishing and Forestry industry even if the direct impacts are very large. Although the recommended allocation for this industry varies with k_1 and $\hat{c}_1^*$, the optimal allocation is less sensitive if $k_1 \geq 0.04$. (Our results initially estimate $k_1 = 0.074$.)

One of the most important parameters in the model is the effectiveness of allocating to all industries, k_0, which determines the amount that should be allocated to stop the oil spill and clean up the the oil. The proportion of resources allocated to all industries in Model 1 is highly sensitive to small changes in k_0 (Figure 4), especially for budgets of $10 billion or less. For example, increasing k_0 from its initial value of $0.74 * 10^{-8}$ to $2.0 * 10^{-8}$ increases the proportion of a $5 billion budget that should be allocated to z_0 from 0.35 to 0.62, and the

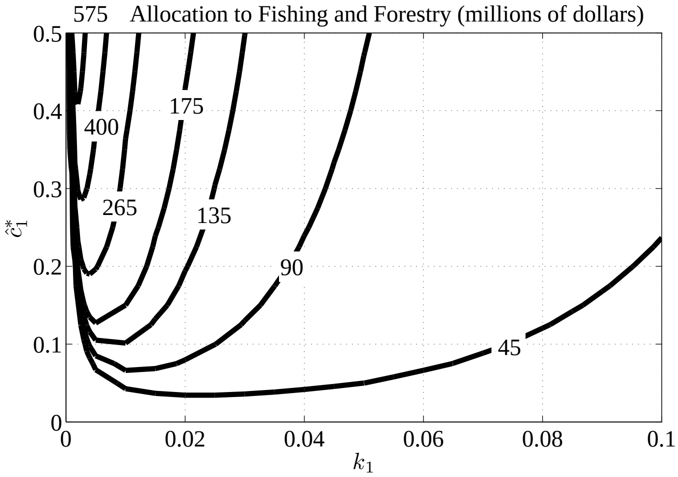

Fig. 3 Contour plots of allocation to Fishing and Forestry industry with a budget of $10 billion

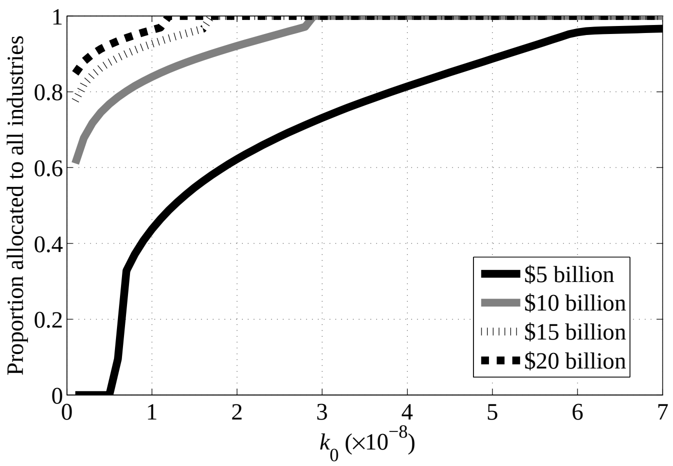

Fig. 4 Sensitivity Analysis on Effectiveness of Allocating Resources to All Industries

proportion for a $10 billion budget increases from 0.81 to 0.92. For values of k_0 greater than $2.0 * 10^{-8}$, the entire budget should be allocated to all industries when the budget is greater than $10 billion. As such, the larger k_0 is, the more effective it is to invest in industry-wide efforts. Because the optimal allocation is highly sensitive to very small changes in k_0, a more careful estimation of this parameter should be undertaken before the model is used as a practical aid in responding to an oil spill.

The model squares the allocation to all industries, z_0, due to the assumption that a lot of resources need to be allocated to all industries before it begins to have a large impact. The results suggest that the vast majority of the budget should be allocated to help all industries for budgets of $10 billion or more. Thus, we examine the extent to which the optimal allocation amounts would change if z_0 were not squared. The power to which z_0 is raised is varied

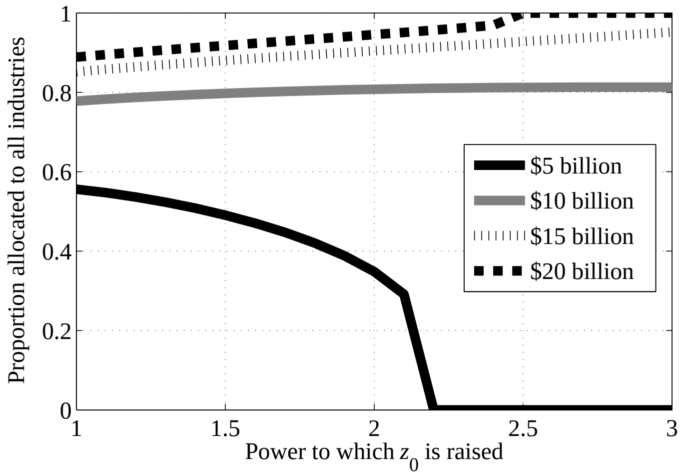

Fig. 5 Sensitivity Analysis on Squaring z_0

from 1 to 3, and Figure 5 records the proportion of the budget allocated to all industries for four different budgets. The value of k_0 also changes so that $k_0^{1/p} * 11600 = 1$ where p is the exponent.

For budgets of \$10 billion or more, the fraction of the budget allocated to all industries remains relatively constant as the power to which z_0 is raised ranges from 1 to 3. A large fraction should be allocated to z_0 partly because of the need to help the Real Estate industry, whose individual allocation effectiveness parameter equals zero. If the budget is \$5 billion, 55% of the budget should be allocated to all industries if z_0 is raised to 1. That percentage decreases as the exponent increases until no money is allocated if the exponent is 2.2 or larger. This sensitivity analysis on a key modeling assumption suggests that for large budgets, a policy maker should allocate the vast majority of the budget to helping all industries even if z_0 is not squared.

5 Conclusion

This paper presented two different models to instruct a policy maker on the effective allocation of resources to help industries recover after a disruption. The first model is a static optimization problem that seeks to minimize production losses as measured by the IIM. The KKT conditions for optimality enable the expression of optimal resource allocations as functions of model parameters, such as the initial impact, the effectiveness of allocating resources, and an industry's production or interdependent effects in an economy. The second model also minimizes production losses but incorporates time by letting the policy maker choose different allocation amounts at different points in time. A branch-and-bound algorithm uses a convex relaxation of the original non-convex dynamic model to provide a lower bound on the minimum value.

Applying these models to the 2010 *Deepwater Horizon* oil spill requires estimating parameters using a variety of newspaper accounts, journal articles, think-tank reports, and government data. If no money is spent on recovery, we estimate that the Gulf region would suffer \$49.1 billion in production losses. The damages from the oil spill were estimated between \$10 and \$20 billion (Aldy 2011), and the Oxford Economics (2010) study proposes that tourism revenues could decline by as much as \$23 billion over a three-year span. If the budget for recovering from the oil spill is \$11.6 billion (the amount that BP spent to stop the spill), the static model estimates that total production losses are \$12.3 billion and losses from the dynamic model total \$10.8 billion (because effectiveness increases with time). These results from the models align closely with the other estimates.

The models in this paper and the application can guide government officials in making decisions about recovering from future disruptions. First, the budget for recovering from a disruption should be large enough to repair physical damage and limit environmental damages. These activities can benefit all of the directly impacted industries simultaneously and may accomplish more than engaging in a risk communication campaign to help specific industries recover. Second, targeting individual industries can be beneficial. It can be optimal to allocate the most money to industries with large direct impacts but where the money is less effective such as Accommodations in the *Deepwater Horizon* application. Finally, when allocation decisions are made over time, unless effectiveness substantially increases with time, it is better to allocate most of the budget in the initial periods although bureaucratic and budgetary obstacles may force policy makers to allocate resources in later periods.

These recommendations do depend on modeling assumptions, however, and policy makers should understand the assumptions behind the models. If the model does not square z_0 or if k_0 is smaller, it may not be optimal to allocate such a large proportion of the budget to all industries. If effectiveness increases over time, or if a dynamic model incorporates uncertainty, it may not be optimal to spend as much as money in the initial time period as suggested by Model 2 in this paper. Although we believe these assumptions are valid for the type of oil spill disruption that motivated this paper, future research should explore the validity of these and other assumptions, such as the exponential effect of resource allocation and the linear model describing interdependent impacts. An empirical study could seek to validate some of these assumptions, or a richer model could be developed that relaxes some of these assumptions and incorporates greater complexity and uncertainty.

Appendix

Proof of Proposition 1

Given a set of $z_i > 0$ and $z_i = 0$, we seek to prove that (6) has at most three solutions. First, we need to prove the following lemma.

Lemma 1 *Given a function $f(x)$ that is continuously differentiable for $x > 0$, if $f'(x) = 0$ has at most two solutions for $x > 0$ and $\alpha > 0$, then $\exp\left(-\alpha x^2\right) f(x) = 0$ has at most three solutions for $x > 0$.*

Proof Assume the function $f(x)$ is continuously differentiable for $x > 0$, $f'(x) = 0$ has at most two solutions, and $\alpha > 0$. If $\exp\left(-\alpha x^2\right) f(x) = 0$ has more than three solutions for $x > 0$, then $f(x) = 0$ has more than three solutions for $x > 0$ because $\exp\left(-\alpha x^2\right) > 0$ for all x.

If $f(x) = 0$ has more than three solutions, then $f(x)$ has at least three local extrema, which means that $f'(x) = 0$ has at least three solutions. This contradicts the premise that $f'(x) = 0$ has at most two solutions. Thus, $\exp\left(-\alpha x^2\right) f(x) = 0$ has at most three solutions for $x > 0$. □

If we can show that the first derivative as depicted in (8) has at most two values for $z_0 > 0$ where the first derivative equals zero, then Lemma 1 applies because (7) is continuously differentiable for $z_0 > 0$.

The second derivative of (7) with respect to z_0 is given in (11).

$$F\exp\left(\frac{z_0-Z}{\sum_{i:z_i>0}1/k_i}\right)\left[\left(\frac{1}{\sum_{i:z_i>0}1/k_i}\right)^2-\frac{2k_0z_0}{\sum_{i:z_i>0}1/k_i}-4k_0\right] \tag{11}$$

Equation (11) has one solution for z_0 if (11) equals zero. This implies that the first derivative in (8) has one extreme point and at most two solutions for z_0 if (8) equals 0. Because no more than two zeros exist for the first derivative in (8), we can conclude from Lemma 1 that (6) has at most three solutions for $0 < z_0 < Z$. □

Proof of Proposition 2

First, we show that if (6) has three real solutions for z_0, then each of the three conditions must hold. If (6) has three solutions, then (7) has three solutions because $\exp\left(-k_0z_0^2\right) \neq 0$. Because the second derivative in (11) has exactly one solution for z_0 when (11) equals 0, the first derivative in (8) has one local extreme point, which means that at most two solutions exist for z_0 when (8) equals 0. This implies that (7) has at most two extreme points. If exactly three solutions exist for z_0 when (7) equals 0, then (7) has a local minimum at $z_0 = z_-$ where (7) is less than 0 and a local maximum at $z_0 = z_+$ where (7) is greater than 0.

Condition 1. The expression in (7) is greater than 0 if $z_0 = 0$. If (7) has three solutions, then (7) must be less than or equal to 0 when $z_0 = Z$. If (7) is less than or equal to 0 when $z_0 = Z$, then $F\left(1-2k_0Z\sum_{i:z_i>0}1/k_i\right)-2Gk_0Z \leq 0$. Then it must be true that $Z \geq F\big/\left(2Gk_0+2Fk_0\sum_{i:z_i>0}1/k_i\right)$, which proves condition 1.

Condition 2. A local minimum exists at $z_0 = z_-$, and a local maximum exists at $z_0 = z_+$. Since (7) is greater than 0 when $z_0 = 0$ and (7) is less than 0 when $z_0 = Z$, it must be true that $z_- < z_+$. The first derivative in (8) equals 0 when $z_0 = z_-$ and $z_0 = z_+$, which implies that $2Gk_0 = F\exp\left(\frac{z_0-Z}{\sum_{i:z_i>0}1/k_i}\right)\left(\frac{1}{\sum_{i:z_i>0}1/k_i}-2k_0z_0-2k_0\sum_{i:z_i>0}\frac{1}{k_i}\right)$ at those two points. Substituting this expression for $2Gk_0$ into (7) leads to the expression in (12).

$$F\exp\left(\frac{z_0-Z}{\sum_{i:z_i>0}1/k_i}\right)\left(2k_0z_0^2-\frac{z_0}{\sum_{i:z_i>0}1/k_i}+1\right) \tag{12}$$

If (12) equals 0, then $z_0 = \left(\frac{1}{\sum_{i:z_i>0}1/k_i} \pm \sqrt{\frac{1}{\left(\sum_{i:z_i>0}1/k_i\right)^2}-8k_0}\right)\bigg/(4k_0)$, which corresponds to z^* and z^{**} as defined in Proposition 2. The expression in (12) is greater than 0 when $z_0 < z^*$ and $z_0 > z^{**}$ and less than 0 when $z^* < z_0 < z^{**}$. If no solution exists when (12) equals 0, then that would imply that (7) is greater than 0 when $z_0 = z_-$ because (12) would always be greater than 0. If that were true, then three solutions would not exist for z_0 when (7) equals 0.

Because (7) is less than 0 when $z_0 = z_-$, then it must be true that $z^* < z_-$. Because (7) is decreasing for $z_0 < z_-$, the first derivative in (8) is less than 0 when $z_0 = z^*$, which proves condition 2.

Condition 3. Because (7) is greater than 0 when $z_0 = z_+$, it must be true that $z^{**} < z_+ < Z$. Because (7) is increasing for $z_- < z_0 < z_+$, (8) is greater than 0 when $z_0 = z_+$, which proves condition 3.

Second, we show that if the three conditions hold, then (6) has three real solutions for z_0. Assume the three conditions hold. Because (8) is less than 0 when $z_0 = z^*$, (7) is decreasing over some range. Because (8) is greater than 0 when $z_0 = z^{**}$, then (7) is increasing over some range. Since $z^* < z^{**}$, there must be a local minimum at some point $z_0 = z_-$ where $z^* < z_- < z^{**}$. The expression in (12) is less than 0 for $z^* < z_0 < z^{**}$, which implies that (7) is less than 0 at the local minimum $z_0 = z_-$. The expression in (7) must equal 0 for some point $0 < z_0 < z_-$.

Since (8) is greater than 0 when $z_0 = z^{**}$, $2Gk_0 < F\exp\left(\frac{z^{**}-Z}{\sum_{i:z_i>0}1/k_i}\right)\left(\frac{1}{\sum_{i:z_i>0}1/k_i}-2k_0z^{**}-2k_0\sum_{i:z_i>0}\frac{1}{k_i}\right)$. This means that (7) is greater than (12) when $z_0 = z^{**}$. Because (12) equals 0 when $z_0 = z^{**}$, (7) is greater than 0. The expression in (7) must equal 0 for some point $z_- < z_0 < z^{**}$.

If $Z \geq \frac{F}{2Gk_0+2Fk_0\sum_{i:z_i>0}1/k_i}$ then (7) is less than or equal to 0 when $z_0 = Z$. Because $z^{**} < Z$ and (7) is greater than 0 when $z_0 = z^{**}$, (7) must equal 0 for some point $z^{**} < z_0 < Z$.

This proves there are at least three solutions for z_0 when (7) equals 0, which means that (6) has at least three solutions for z_0. From Proposition 1, no more than three solutions exist, and we conclude that (6) has three solutions for z_0. □

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