We proceed to define fitted effects for 3- (and by analogy, higher-) way studies, beginning with **fitted main effects**. These are

\[ a_i = \bar{y}_{i.} - \bar{y} \]
\[ = (\text{the Factor A level } i \text{ average } \bar{y}) - (\text{the grand average } \bar{y}) \]
\[ = \text{the (fitted) main effect of the } i\text{th level of Factor A} \]

\[ b_j = \bar{y}_{.j} - \bar{y} \]
\[ = (\text{the Factor B level } j \text{ average } \bar{y}) - (\text{the grand average } \bar{y}) \]
\[ = \text{the (fitted) main effect of the } j\text{th level of Factor B} \]

and

\[ c_k = \bar{y}_{..k} - \bar{y} \]
\[ = (\text{the Factor C level } k \text{ average } \bar{y}) - (\text{the grand average } \bar{y}) \]
\[ = \text{the (fitted) main effect of the } k\text{th level of Factor C} \]
Main effects

Because it will be useful in discussing the meaning of 3-way factorial effects, we repeat here the cube plot of means presented in the last module.

\[
\begin{align*}
\bar{y}_b &= 54 \\
\bar{y}_c &= 52 \\
\bar{y}_a &= 72 \\
\bar{y}_{bc} &= 45 \\
\bar{y}_{ab} &= 68 \\
\bar{y}_{ac} &= 83 \\
\bar{y}_{abc} &= 80
\end{align*}
\]
Defining 3-Way (and Higher-Way) Fitted Factorial Effects

Main effects

Main effects in a $2^3$ study are differences between "face" average means and the grand average mean on a plot like that on panel 3. As portrayed on that panel

\[ a_2 = (\text{the right face average } \bar{y}) - (\text{the grand average } \bar{y}) \]

and

\[ b_2 = (\text{the top face average } \bar{y}) - (\text{the grand average } \bar{y}) \]

and

\[ c_2 = (\text{the back face average } \bar{y}) - (\text{the grand average } \bar{y}) \]

Just as in 2-way studies, the fitted main effects for any factor add to 0 over levels of that factor (so that a fitted main effect for one level in a $2^p$ study is just minus one times that for the other level).
In the pilot plant study, it is straightforward to see that

\[ a_1 = 52.75 - 64.25 = -11.5 \quad \text{and} \quad a_2 = 75.75 - 64.25 = 11.5 \]

and

\[ b_1 = 66.75 - 64.25 = 2.5 \quad \text{and} \quad b_2 = 61.75 - 64.25 = -2.5 \]

and

\[ c_1 = 63.5 - 64.25 = -0.75 \quad \text{and} \quad c_2 = 65.0 - 64.25 = 0.75 \]

The relative sizes of the A, B, and C fitted main effects quantify what is to some extent already obvious on panel 3: The left-to-right differences in means on the corners of the cube are bigger than the top-to-bottom or front-to-back differences.
Fitted 2 factor interactions in a 3-way factorial can be thought of in at least two ways. First, they are measures of lack of parallelism that would be appropriate after averaging out over levels of the 3rd factor. Second, they represent what can be explained about a response mean if one thinks of factors acting jointly in pairs beyond what is explainable in terms of them acting separately. The definitions of these look exactly like the definitions from two-factor studies, except that an extra dot appears on each \( \bar{y} \). That is

\[
ab_{ij} = \bar{y}_{ij} - (\bar{y} + a_i + b_j)
\]

and

\[
ac_{ik} = \bar{y}_{i.k} - (\bar{y} + a_i + c_k)
\]

and

\[
b_{ck} = \bar{y}_{.jk} - (\bar{y} + b_j + c_k)
\]
It is a consequence of their definitions that fitted 2-factor interactions add to 0 over levels of any one of the factors involved. In a $2^p$ study, this allows one to compute a single one of these fitted interactions of a given type and obtain the other three by simple sign changes. For example

$$ab_{11} = -ab_{12} = -ab_{21} = ab_{22}$$

(if the number of "index-switches" going from one set of indices to another is odd, the sign of the fitted effect changes, while if the number is even there is no sign change).
In the pilot plant study,

\[
ab_{11} = \bar{y}_{11} - (\bar{y}... + a_1 + b_1) \\
= 56 - (64.25 + (-11.5) + 2.5) = .75
\]

so that

\[
ab_{12} = -.75 \quad \text{and} \quad ab_{21} = -.75 \quad \text{and} \quad ab_{22} = .75
\]

Similarly,

\[
ac_{11} = \bar{y}_{1.1} - (\bar{y}... + a_1 + c_1) \\
= 57 - (64.25 + (-11.5) + (-.75)) = 5.0
\]

so that

\[
ac_{12} = -5.0 \quad \text{and} \quad ac_{21} = -5.0 \quad \text{and} \quad ac_{22} = 5.0
\]
And finally,

\[ bc_{11} = \bar{y}_{..} - (\bar{y}_{..} + b_1 + c_1) \]
\[ = 66 - (64.25 + 2.5 + (-.75)) = 0 \]

so that

\[ bc_{12} = 0 \quad \text{and} \quad bc_{21} = 0 \quad \text{and} \quad bc_{22} = 0 \]

This last set of values says, for example, that after averaging over levels of Factor A, there would be perfect parallelism on an interaction plot of means \( \bar{y}_{jk} \). On the other hand, the fairly large size of the AC two-factor interactions is consistent with the clear lack of parallelism on the figure on panel 10, that is an interaction plot of averages of \( \bar{y} \)'s top to bottom (over levels of Factor B) from the figure on panel 3.
Fitted 2-Factor Interactions

Example 46-1 continued

Figure: Interaction Plot for Pilot Plant Study After Averaging over Levels of B (Showing Strong AC Two-Factor Interaction/Lack of Parallelism)
Fitted 3 factor interactions in a three way study are defined as

\[ abc_{ijk} = \bar{y}_{ijk} - (\bar{y}_{\ldots} + a_i + b_j + c_k + ab_{ij} + ac_{ik} + bc_{jk}) \]

These measure the difference between what’s observed and what’s explainable in terms of factors acting separately and in pairs on the response, \( y \). They are also the difference between what one would call 2 factor interactions between, say, A and B, looking separately at the various levels of C (so that they are 0 exactly when the pattern of AB two factor interaction is the same for all levels of C).

These 3 factor interactions sum to 0 over all levels of any of the factors A, B, and C, so for a \( 2^p \) factorial one may compute one of these and get the others by appropriate sign changes.
In the pilot plant study,

\[ abc_{111} = \bar{y}_{111} - (\bar{y}_{...} + a_1 + b_1 + c_1 + ab_{11} + ac_{11} + bc_{11}) \]
\[ = 60 - (64.25 + (-11.5) + 2.5 + (-.75) + .75 + 5.0 + 0) = -.25 \]

so that

\[ abc_{112} = .25, abc_{121} = .25, abc_{211} = .25, \]
\[ abc_{122} = -.25, abc_{212} = -.25, abc_{221} = -.25, \] and \[ abc_{222} = .25 \]

Happily, these (relatively difficult to interpret, at least as compared to main effects and 2 factor interactions) fitted effects are small in comparison to the lower order fitted effects.