IE 361 Module 4
Modeling Measurement

Reading: Section 2.1 *Statistical Methods for Quality Assurance*

ISU and Analytics Iowa LLC
A basic statistical/probabilistic model for measurement is that what is measured, $y$, is the *measurand*, $x$, plus a normal random measurement error, $\epsilon$, with mean $\delta$ and standard deviation $\sigma_{\text{device}}$.

$$y = x + \epsilon$$

Pictorially this is as in the following figure.
A Simple Measurement Model

Figure: A Basic Probability Model for Measurement
The difference between the measurand and the mean measurement is the bias in measurement, $\delta$. Ideally, this is 0 (and it is the business of calibration to attempt to make it 0). At a minimum, measurement devices are designed to have a "linearity" property. This means that that over the range of measurands a device will normally be used to evaluate, if its bias is not 0, it is at least constant (i.e. $\delta$ does not depend upon $x$). This is illustrated in cartoon form in the next figure (where we assume that the vertical and horizontal scales are the same).
A Simple Measurement Model

The line $\mu_y = x + \delta$

constant bias, $\delta$

the line $\mu_y = x$

range of device use

Figure: A Cartoon Illustrating Measurement Device "Linearity"
A Simple Measurement Model

When one makes multiple measurements on the same measurand using a fixed device, one has not a single, but multiple data values, $y_1, y_2, \ldots, y_n$. It is standard to model these repeat measurements as independent random draws from the same distribution (with mean $x + \delta$ and standard deviation $\sigma_{\text{device}}$) illustrated on panel 3. We will abbreviate this model assumption as

$$y_i \sim \text{ind} \ (x + \delta, \sigma_{\text{device}})$$

and illustrate it in the following figure.
A Simple Measurement Model

Figure: Cartoon Illustrating Multiple Measurements on a Single Measurand

$$y_i's \sim \text{ind} \left( x + \delta, \sigma_{\text{device}} \right)$$
A Simple Measurement Model

Note in passing (we’ll have more to say about this directly) that this figure makes clear that the sample mean from repeat measurements of this type, \( \bar{y} \), can be expected to approximate \( x + \delta \) (the measurand plus bias). Further, the sample standard deviation, \( s \), can be expected to approximate the device standard deviation, \( \sigma_{device} \).

Most often in quality assurance applications, there is not a single measurand, but rather multiple measurands representing some important feature of multiple items or batches produced by a production process. In this context, we extend the basic measurement model by assuming that \( x \) varies/is random. (Variation in \( x \) is "real" process variation, not just measurement variation.) In fact, if the device is linear and the measurand is itself normal with mean \( \mu_x \) and standard deviation \( \sigma_x \) and independent of the measurement error, we then have

\[
y = x + \epsilon
\]
with mean

$$\mu_y = \mu_x + \delta$$

and standard deviation

$$\sigma_y = \sqrt{\sigma_x^2 + \sigma_{\text{device}}^2} > \sigma_x$$

(so observed variation in $y$ is larger than the process variation because of measurement noise). Notice that these relationships are consequences of the basic (Stat 231) facts that if $U$ and $V$ are independent random variables $\mu_{U+V} = \mu_U + \mu_V$ and $\sigma_{U+V}^2 = \sigma_U^2 + \sigma_V^2$. (These are the "laws of expectation (or mean) and variance." If they don’t look completely familiar, you should find and review them in your basic statistics text.)
A Simple Measurement Model

It is quite common in quality assurance applications to select $n$ items from the output of a process and make a single measurement on each, with the intention of drawing inferences about the process. If the process can be thought of as physically stable and the device is linear, it is standard to model these measurements as *independent random draws from the same distribution* (with mean $\mu_x + \delta$ and standard deviation $\sqrt{\sigma_x^2 + \sigma_{\text{device}}^2}$). We can abbreviate this model assumption as

$$y_i \sim \text{ind} \left( \mu_x + \delta, \sqrt{\sigma_x^2 + \sigma_{\text{device}}^2} \right)$$

and illustrate it in the following figure.
A Simple Measurement Model

\[ \mu_x, \sigma_x \]

\[ x_1 \quad x_2 \quad \cdots \quad x_n \]

\( \delta, \sigma_{\text{device}} \)

\[ \begin{array}{c}
\cdots \\
y_1 \\
y_2 \\
\vdots \\
y_n
\end{array} \]

\[ y_i's \sim \text{ind} \left( \mu_x + \delta, \sqrt{\sigma_x^2 + \sigma_{\text{device}}^2} \right) \]

**Figure:** Cartoon Illustrating Single Measurements on Multiple Items From a Stable Process (Assuming Device Linearity)
Panels 7 and 11 are *not* the same, even though they both lead to a "sample" of \( n \) measurements \( y \). Here, \( \bar{y} \), can be expected to approximate \( \mu_x + \delta \) (the process mean plus bias) while the sample standard deviation, \( s \), can be expected to approximate a combination of the device standard deviation and the process standard deviation.

A comparison of panels 7 and 11 illustrates the very important and basic insight that:

> How sources of physical variation interact with a data collection plan governs what of practical importance can be learned from a data set, and in particular, how measurement error is reflected in the data set.

This principle governs the practical use of even the very simplest of statistical methods. We begin to illustrate this in following modules.