IE 361 Module 38
Process Capability Analysis Part 3 (Prediction and Tolerance Intervals)

Reading: Section 4.3 Statistical Methods for Quality Assurance

ISU and Analytics Iowa LLC
"Capability" and Future Values

The measures $6\sigma$, $C_p$, and $C_{pk}$ considered in Module 37 attempt to summarize process "capability" in terms of a function of process parameters (and where appropriate, specifications for individual outcomes). In this module we consider methods of characterizing process output that focus what can be said about the values of individual future process outcomes based on data in hand (rather than about process summary measures).
If I KNOW process parameters, making statements about future individual values generated by the process is a matter of simple probability calculation. Suppose, for example, that I model individual values as normal with $\mu = 7$ and $\sigma = 1$.

- there’s a "90% chance" the next $x$ is between 5.355 and 8.645
- 90% of the process distribution is between 5.355 and 8.645

But what if I only have a sample, and not the process parameters? What then can I say? When one has to use a sample to get an approximate picture of a process, it is important to hedge statements in light of sample variability/uncertainty . . . this can be done

- for normal processes using $\bar{x}$ and $s$
- in general, using the sample minimum and/or maximum values
Methods for Normal Processes

Prediction Limits

We consider first methods for normal processes. (Just as we cautioned in Module 37 that the methods for estimating capabilities are completely unreliable unless the data-generating process is adequately described by a normal model, so too does the effectiveness of the next 2 formulas depend critically on the normal assumption being appropriate.)

For normal processes, "prediction limits" for a single additional individual are

\[ \bar{x} \pm ts\sqrt{1 + \frac{1}{n}} \]

Prediction limits are sometimes met in the context of regression analysis, but the simple one-sample limits above are even more basic and unfortunately not always taught in an introductory course. They are intended to capture a single additional observation from the process that generated \( \bar{x} \) and \( s \).
Methods for Normal Processes

Example 38-1

(EDM drilling) What do the \( n = 50 \) angles in Table 4.7 say about additional angles drilled by the process? (Recall that here \( \bar{x} = 44.117^\circ \) and \( s = .984^\circ \).) One answer can be phrased in terms of a 95% prediction interval for a single additional output. Using the fact that the upper 2.5% point of the \( t \) distribution for \( df \nu = 50 - 1 = 49 \) is 2.010, 95% prediction limits for a single additional output are

\[
44.117 \pm 2.010 (.984) \sqrt{1 + \frac{1}{50}} \quad \text{i.e.} \quad 44.117 \pm 1.998
\]

One can in some sense be 95% sure that the next angle drilled will be between \( 42.119^\circ \) and \( 46.115^\circ \).

The 95% confidence value is a "lifetime batting average" associated with a long series of repetitions of the whole business of selecting \( n \), making the interval, selecting one more value, and checking for success. (In any given application of the method one is either 100% right or 100% wrong.)
Methods for Normal Processes
Tolerance Limits

Another way to identify what to expect from future process outcomes might be to locate not just a single outcome, but some large fraction \((p)\) of all future outcomes (under the current process conditions). For a normal process, **two-sided "tolerance limits" for a large fraction \((p)\) of all additional individuals** are

\[
\bar{x} \pm \tau_2 s
\]

(the values \(\tau_2\) are special constants tabled in Table A.6.1 of *SMQA*). (One-sided limits are similar, but use constants \(\tau_1\) from Table A.6.2 of *SMQA*.) Tolerance limits are not always taught in an introductory statistics course, so some students will not have seen this idea before.
A second answer to the question "What do the $n = 50$ angles in Table 4.7 say about additional angles drilled by the process?" can be phrased in terms of a 99% tolerance interval for 95% of all values from the process. (This would be an interval that one is "99% sure" contains "95% of all future values.") Reading directly in Table A.6.1 of SMQA produces a multiplier of $\tau_2 = 2.58$ (note that the table is set up in terms of sample size, NOT degrees of freedom) and therefore the two-sided tolerance limits

$$44.117 \pm 2.58(0.984) \text{ or } 44.117^\circ \pm 2.54^\circ$$

for the bulk of all future angles (assuming, of course, that current process conditions are maintained into the future). (A one-sided tolerance limit can be had by replacing $\tau_2$ with a value $\tau_1$ from Table A.6.2 of SMQA.)
A "thought experiment" illustrating the meaning of "confidence" associated with a tolerance interval method involves:

1. drawing multiple samples,
2. for each one computing the limits $\bar{x} \pm \tau_2 s$,
3. for each one using a normal distribution calculation based on the true process parameters to ascertain the fraction of the population covered by the sample interval, and
4. checking to see if the fraction in 3 is at least the desired value $p$ ... if it is, the interval is a success, if it is not, the interval is a failure.

The confidence level for the method is then the lifetime batting average of the method.
A second approach to making prediction and tolerance intervals (that doesn’t depend upon normality of the data-generating process for its validity, only on process stability) involves simply using the smallest and largest data values in hand to state limits on future individuals. That is, one may use the interval

\[(\min x_i, \max x_i)\]

as either a prediction interval or a tolerance interval. Provided the "random sampling from a fixed universe" model is sensible,

- used as a **prediction interval for one more observation** this has confidence level
  \[
  \frac{n - 1}{n + 1}
  \]

- used as a **tolerance interval for a fraction \(p\) of all future observations** from the process it has associated confidence level
  \[
  1 - p^n - n (1 - p) p^{n-1}
  \]
The smallest and largest angles among the \( n = 50 \) in the data set of Table 4.7 are respectively 42.017 and 46.050. We consider the interval

\[(42.017, 46.050)\]

for locating future observed angles.

As a prediction interval for the next one, the appropriate confidence level is

\[
\frac{50 - 1}{50 + 1} = .961 = 96.1\%
\]

And, for example, as a tolerance for 95% of EDM drilled angles, the appropriate confidence level is

\[
1 - (.95)^{50} - 50 (.05) (.95)^{49} = .721 = 72.1\%
\]
The interpretation of confidence level associated with these intervals based on sample minimum and maximum values is exactly the same as for the normal distribution prediction and tolerance intervals based on $\bar{x}$ and $s$. The news here is that these levels are guaranteed for any (continuous) process distribution ... normal or not.

We should also note that it is possible to employ only one of the values min $x_i$ and max $x_i$ to make one-sided prediction and tolerance intervals. The associated confidence levels are given in the summary table on panel 12, and are (of course) larger than those for the (smaller) two-sided intervals.
<table>
<thead>
<tr>
<th>Two-Sided Intervals</th>
<th>One-Sided Intervals</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PI</strong> ( \left( x - ts \sqrt{1 + \frac{1}{n}}, x + ts \sqrt{1 + \frac{1}{n}} \right) )</td>
<td>( \left( x - ts \sqrt{1 + \frac{1}{n}}, \infty \right) )</td>
</tr>
<tr>
<td><strong>TI</strong> ( (x - \tau_2 s, x + \tau_2 s) )</td>
<td>( (x - \tau_1 s, \infty) )</td>
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</tbody>
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<table>
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<tr>
<th>Any Stable Process</th>
<th>( \left( \min x_i, \max x_i \right) ) (\text{confidence} = \frac{n-1}{n+1} )</th>
<th>( \left( \min x_i, \infty \right) ) or ( (-\infty, \max x_i) ) (\text{confidence} = \frac{n}{n+1} )</th>
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<td><strong>TI</strong> ( \left( \min x_i, \max x_i \right) ) (\text{confidence} = \frac{1 - p^n - n(1-p)p^{n-1}}{1 - p^n} )</td>
<td>( \left( \min x_i, \infty \right) ) or ( (-\infty, \max x_i) ) (\text{confidence} = \frac{1 - p^n}{1 - p^n} )</td>
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**Figure:** Prediction and Tolerance Interval Summary