IE 361 Module 29

Control Charts for Counts (Attributes Data) Part 1

Reading: Section 3.3 Statistical Methods for Quality Assurance

ISU and Analytics Iowa LLC
In this module and the next, we discuss the Shewhart control charts for so-called "fraction nonconforming" and "mean nonconformities per unit" contexts. These are the Shewhart $p$ and $np$ charts and the Shewhart $u$ (and $c$) charts. These tools are easy enough to explain and use, but are typically really NOT very effective in modern applications where acceptable nonconformity rates are often so small as to be stated in "parts per million."
Control Charts for Fraction Nonconforming

The scenario under which a $p$ chart or ($np$ chart) is potentially appropriate is one where periodically groups of $n$ items (or outcomes) from a process are looked at and

$$X = \text{the number of outcomes among the } n \text{ that are "nonconforming"}$$

is observed. This is illustrated in the figure below, where dark balls represent nonconforming outcomes.

**Figure:** Cartoon Representing $n$ Process Outcomes, $X$ of Which are Nonconforming
Probability Basis for Control Limits

In this kind of circumstance, the notation

\[ \hat{p} = \frac{X}{n} = \text{the sample fraction nonconforming} \]

is standard, and

- control charts for \( \hat{p} \) are called \( p \) charts
- control charts for \( X \) (\( = n\hat{p} \)) are called \( np \) charts

If the process producing items/outcomes is physically stable, a reasonable probability model for \( X \) (met in Stat 231) is the binomial \( (n, p) \) distribution, where

\[ p = \text{the current probability that any particular outcome is nonconforming} \]

(the mental fiction here is that the particular \( n \) outcomes observed are a random sample of a huge pool of outcomes, a fraction \( p \) of which are nonconforming).
Standards Given Control Limits

Stat 231 facts about the binomial distribution are that

$$\mu_X = np \quad \text{and} \quad \sigma_X = \sqrt{np(1-p)}$$

so that (since $\hat{p} = \left(\frac{1}{n}\right)X$)

$$\mu_{\hat{p}} = p \quad \text{and} \quad \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$

These facts in turn lead to **standards given (np chart)** control limits for $X$

$$LCL_X = np - 3\sqrt{np(1-p)} \quad \text{and} \quad UCL_X = np + 3\sqrt{np(1-p)}$$

and **standards given (p chart)** control limits for $\hat{p}$

$$LCL_{\hat{p}} = p - 3\sqrt{\frac{p(1-p)}{n}} \quad \text{and} \quad UCL_{\hat{p}} = p + 3\sqrt{\frac{p(1-p)}{n}}$$
Below are some artificially generated (using \( p = .03 \)) \( n = 400 \) binomial data (except for sample 9, where a larger value of \( p \) was used) and the corresponding values of \( \hat{p} \).

<table>
<thead>
<tr>
<th>Sample</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X )</td>
<td>15</td>
<td>11</td>
<td>18</td>
<td>9</td>
<td>13</td>
<td>11</td>
<td>10</td>
</tr>
<tr>
<td>( \hat{p} )</td>
<td>.0375</td>
<td>.0275</td>
<td>.0450</td>
<td>.0225</td>
<td>.0325</td>
<td>.0275</td>
<td>.0250</td>
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</table>

<table>
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<tr>
<th>Sample</th>
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<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X )</td>
<td>19</td>
<td>24</td>
<td>7</td>
<td>9</td>
<td>13</td>
<td>17</td>
<td>7</td>
</tr>
<tr>
<td>( \hat{p} )</td>
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<td>.0600</td>
<td>.0175</td>
<td>.0225</td>
<td>.0325</td>
<td>.0425</td>
<td>.0175</td>
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</table>

<table>
<thead>
<tr>
<th>Sample</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X )</td>
<td>10</td>
<td>19</td>
<td>11</td>
<td>8</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>( \hat{p} )</td>
<td>.0250</td>
<td>.0475</td>
<td>.0275</td>
<td>.0200</td>
<td>.0200</td>
<td>.0175</td>
</tr>
</tbody>
</table>
Standards given control limits for $\hat{p}$ here are

$$LCL_{\hat{p}} = p - 3\sqrt{\frac{p(1-p)}{n}} = .03 - 3\sqrt{\frac{.03(1-.03)}{400}} = .0044$$

and

$$UCL_{\hat{p}} = .03 + 3\sqrt{\frac{.03(1-.03)}{400}} = .0556$$
A plot of the corresponding control chart is in the figure below and shows clearly that sample 9 produces an out-of-control signal. That sample simply does not fit the "stable process with $p = .03$" model that stands behind the control limits.

**Figure:** Standards Given ($p = .03$) $p$ Chart for the Artificial Data
Retrospective control limits for $X$ or $\hat{p}$ are obtained (as always) by making a provisional assumption of process stability and estimating process parameters (here, the value $p$). The most sensible estimate obtainable from $r$ values $X_i = n\hat{p}_i$ based on respective sample sizes $n_i$ is

$$\hat{p}_{\text{pooled}} = \frac{\text{total outcomes nonconforming}}{\text{total outcomes observed}} = \frac{X_1 + X_2 + \cdots + X_r}{n_1 + n_2 + \cdots + n_r}$$

(This is the arithmetic mean of the $\hat{p}_i$ in cases where the sample size is constant.)
Returning to the artificial data on panel 6, there are a total of 246 nonconforming outcomes indicated among the 20 $(400) = 8000$ outcomes represented. So

\[
\hat{p}_{\text{pooled}} = \frac{246}{8000} = .0308
\]

Retrospective control limits for \( \hat{p} \) are thus

\[
LCL_{\hat{p}} = .0308 - 3\sqrt{\frac{.0308 (1 - .0308)}{400}} = .0049
\]

and

\[
UCL_{\hat{p}} = .0308 + 3\sqrt{\frac{.0308 (1 - .0308)}{400}} = .0567
\]
The figure below shows that the retrospective limits do not produce a picture much different from the standards given ones used to make panel 8. The 20 samples do not fit with any "constant p" model.

Figure: Retrospective $p$ Chart for the Artificial Data
Notice that a lower control limit for a $p$ chart CAN be positive. (Example 29-1 shows this.) This sometimes seems counter-intuitive, but only to those who fail to remember that control charts are about detecting process change, NOT about product acceptability. When a point plots below a lower control limit on a $p$ or $np$ chart, one is alerted to unexpected/fortuitous process improvement. One wants to follow up on such lucky circumstances, looking for an assignable cause to incorporate into the process on a permanent basis.
Varying Sample Size

In some cases, sample sizes vary period to period. When that happens, each new period may need a new calculation of control limits for that sample size. NOTICE that big sample sizes carry relatively larger amounts of information about current process conditions and have control limits for \( \hat{p} \) that are tighter around a center line than small sample sizes. (With more information in a sample, less deviation from the expected value for \( \hat{p} \) is required before one is fairly sure that something has changed in the process.)

Example 29-2  Below is a comparison between \( (p = .03) \) standards given limits for \( \hat{p} \) for \( n = 400 \) and \( n = 100 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( CL_{\hat{p}} )</th>
<th>( LCL_{\hat{p}} )</th>
<th>( UCL_{\hat{p}} )</th>
</tr>
</thead>
<tbody>
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<td>none</td>
<td>.0812</td>
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<tr>
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<td>.03</td>
<td>.0044</td>
<td>.0556</td>
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