IE 361 Module 13
The Two Way Random Effects Model and Gauge R&R

Reading: Section 2.4.1 Statistical Methods for Quality Assurance

ISU and Analytics Iowa LLC
The "Two-Way Random Effects" Model for Gauge R&R Data

Typical analyses of Gauge R&R studies are based on the so-called "two-way random effects" model. With

\[ y_{ijk} = \text{the } k\text{th measurement made by operator } j\text{ on specimen } i \]

this model is that \( y_{ijk} \) is made up as a sum of independent contributions,

\[ y_{ijk} = \mu + \alpha_i + \beta_j + \alpha \beta_{ij} + \epsilon_{ijk} \]

where

- \( \mu \) is an (unknown) constant, an average (over all possible operators and all possible parts/specimens) measurement
- the \( \alpha_i \) are normal with mean 0 and variance \( \sigma^2_{\alpha} \), (random) effects of different parts/specimens
- the \( \beta_j \) are normal with mean 0 and variance \( \sigma^2_{\beta} \), (random) effects of different operators
The Two-Way Random Effects Model

- the $\alpha_{ij}$ are normal with mean 0 and variance $\sigma^2_{\alpha \beta}$, (random) joint effects peculiar to particular part/operator combinations
- the $\epsilon_{ijk}$ are normal with mean 0 and variance $\sigma^2$, (random) errors that are peculiar to a particular attempt to make a measurement (they change measurement-to-measurement, even if the part and operator remain the same)

$\sigma^2_{\alpha}$, $\sigma^2_{\beta}$, $\sigma^2_{\alpha \beta}$, and $\sigma^2$ are called "variance components" and their sizes govern how much variability is seen in the measurements $y_{ijk}$. 
Example 13-1

The reader should conduct a "Thought Experiment" generating a Gauge R&R data set, and fill in formulas for the 12 measurements in the table below. (For example, $y_{111} = \mu + \alpha_1 + \beta_1 + \alpha\beta_{11} + \epsilon_{111}$.)

<table>
<thead>
<tr>
<th>Part</th>
<th>Operator 1</th>
<th>Operator 2</th>
<th>Operator 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$y_{111} =$</td>
<td>$y_{121} =$</td>
<td>$y_{131} =$</td>
</tr>
<tr>
<td></td>
<td>$y_{112} =$</td>
<td>$y_{122} =$</td>
<td>$y_{132} =$</td>
</tr>
<tr>
<td>2</td>
<td>$y_{211} =$</td>
<td>$y_{221} =$</td>
<td>$y_{231} =$</td>
</tr>
<tr>
<td></td>
<td>$y_{212} =$</td>
<td>$y_{222} =$</td>
<td>$y_{232} =$</td>
</tr>
</tbody>
</table>
The Two-Way Random Effects Model

In this (two-way random effects) model

- $\sigma$ measures within-cell/repeatability variation
- $\sigma_{\text{reproducibility}} = \sqrt{\sigma_\beta^2 + \sigma_{\alpha\beta}^2}$ is the standard deviation that would be experienced by many operators measuring the same specimen once each, \textit{in the absence of repeatability variation}
- $\sigma_{R&R} = \sqrt{\sigma_{\text{reproducibility}}^2 + \sigma^2} = \sqrt{\sigma_\beta^2 + \sigma_{\alpha\beta}^2 + \sigma^2}$ is the standard deviation that would be experienced by many operators measuring the same specimen once each (this is called $\sigma_{\text{overall}}$ in \textit{SQAME})

To make connections to what we have done earlier, consider what these two-way model parameters mean if we restrict attention to part #1. The two-way random effects model says that measurements on part #1 can be thought of as

$$y_{1jk} = \mu + \alpha_1 + \beta_j + \alpha\beta_{1j} + \epsilon_{1jk}$$
The Two-Way Random Effects Model

What then varies operator-to-operator is

$$\beta_j + \alpha \beta_1 j$$

This quantity thus plays the role of what we before called $\delta_j$ (operator bias for operator $j$ ... for part #1) and

$$\sigma^2_{\beta_j + \alpha \beta_1 j} = \sigma^2_{\beta} + \sigma^2_{\alpha \beta}$$

plays the role of what we before called $\sigma^2_\delta$ (the reproducibility variance). The fact that $\beta_j + \alpha \beta_1 j$ is specific to part #1 (for example changes to $\beta_j + \alpha \beta_2 j$ if part #2 is considered instead) has the interesting interpretation that the terms $\alpha \beta_{ij}$ play the role of "device" nonlinearities! That is, in the two-way random effects model where multiple parts are considered, a large variance component $\sigma^2_{\alpha \beta}$ is indicative of substantially non-constant bias of the various operators in their use of the gauge ... a most unpleasant circumstance indeed.
The Two-Way Random Effects Model

The most common analyses (both those based on ranges and those based on ANOVA) (e.g. following the AIAG manual) are wrong, in that they purport to produce estimates of $\sigma_{\text{reproducibility}}$ and $\sigma_{R&R}$ but fail to do so. SQAME and the next module present correct range-based and ANOVA-based methods. We will use primarily the generally more effective ANOVA-based estimates and confidence intervals that can be based on them (these limits are not found in SQAME).

If you end up doing a gauge R&R study for your project client, you will almost certainly be asked to use company standard formulas or a company spreadsheet that implements the (WRONG) AIAG formulas. If you do this you should compare those results to ones obtained using correct formulas from these modules! Failure to do so will be frowned upon when projects are graded.