IE 361 Handout: Deriving Estimates of Types of Variability

Michael Dorneich, 27 January 2015

The ANOVA identity is given by \( SSTot = SSTR + SSE \), or

\[
\sum_{i=0}^{I} \sum_{j=0}^{n_i} (y_{ij} - \bar{y})^2 = \sum_{i=0}^{I} n_i (\bar{y}_i - \bar{y})^2 + \sum_{i=0}^{I} \sum_{j=0}^{n_i} (y_{ij} - \bar{y}_i)^2
\]

Where

- \( I \) = number of students
- \( n_1 = n_2 = \ldots = n_I = m = \# \text{ measurements / student} \)
- \( n = I \times m \)

**Derivation of a Method for Calculating SSTot**

\[
SSTot = \sum_{i=0}^{I} \sum_{j=0}^{n_i} (y_{ij} - \bar{y})^2
\]

\[
SSTot = \sum_{i=0}^{I} \sum_{j=0}^{n_i} (y_{ij} - \bar{y})^2 \frac{(n-1)}{(n-1)} = (n-1) \frac{\sum_{i=0}^{I} \sum_{j=0}^{n_i} (y_{ij} - \bar{y})^2}{(n-1)}
\]

The **sample variance of all measurements** is given by

\[
\bar{s}^2 = \frac{\sum_{i=0}^{I} \sum_{j=0}^{n_i} (y_{ij} - \bar{y})^2}{n-1}
\]

Thus by simple substitution we have

\[
SSTot = (n-1)\bar{s}^2
\]
Derivation of a Method for Calculating SSE

The variance of any one operator is given by

$$s_i^2 = \frac{1}{(m - 1)} \sum_{j=0}^{n} (y_{ij} - \bar{y}_i)^2$$

So inserting this into the equation above results in

$$SSE = \sum_{i=0}^{1} \sum_{j=0}^{n} (y_{ij} - y_i)^2 = (m - 1) \sum_{i=0}^{1} \sum_{j=0}^{n} (y_{ij} - y_i)^2$$

The variance of any one operator is given by

$$s_i^2 = \frac{1}{(m - 1)} \sum_{j=0}^{n} (y_{ij} - \bar{y}_i)^2$$

So inserting this into the equation above results in

$$SSE = (m - 1) \sum_{i=0}^{1} \sum_{j=0}^{n} (y_{ij} - y_i)^2 = (m - 1) \sum_{i=0}^{1} s_i^2$$

But we also know that if we take the arithmetic average of the sample variance per operator (below):

$$\bar{s}_i^2 = \frac{1}{l} (s_1^2 + s_2^2 + \cdots + s_l^2) = \frac{1}{l} \sum_{i=0}^{l} s_i^2$$

Then by simple substitution we have

$$SSE = l(m - 1) \frac{1}{l} \sum_{i=0}^{l} s_i^2 = l(m - 1) \bar{s}_i^2$$
Derivation of a Method for Calculating SSTr

First of all, if you know SSTot and SSE, then SSTr = SSTot - SSE.

You can also derive a method for calculating it as follows:

\[
SSTr = \sum_{i=0}^{I} n_i (\bar{y}_i - \bar{y})^2
\]

\[
SSTr = \sum_{i=0}^{I} n_i (\bar{y}_i - \bar{y})^2 = \frac{I-1}{I-1} \sum_{i=0}^{I} n_i (\bar{y}_i - \bar{y})^2 = (I-1) \sum_{i=0}^{I} n_i \frac{1}{I-1} \sum_{i=0}^{I} n_i (\bar{y}_i - \bar{y})^2
\]

The sample variance of student means is given by

\[
\bar{s}_{\bar{y}}^2 = \frac{1}{I-1} \sum_{i=0}^{I} (\bar{y}_i - \bar{y})^2
\]

and

\[
\sum_{i=0}^{I} n_i = m
\]

By simple substitution we have

\[
SSTr = (I-1) \sum_{i=0}^{I} n_i \frac{1}{I-1} \sum_{i=0}^{I} n_i (\bar{y}_i - \bar{y})^2 = (I-1) m \bar{s}_{\bar{y}}^2
\]